

Weak attractor of the Klein-Gordon field in discrete space-time interacting with a nonlinear oscillator

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Abstract

We consider the $U(1)$ -invariant nonlinear Klein-Gordon equation in discrete space and discrete time, which is the discretization of the nonlinear continuous Klein-Gordon equation. To obtain this equation, we use the energy-conserving finite-difference scheme of Strauss-Vazquez. We prove that each finite energy solution converges as $T \rightarrow \pm\infty$ to the finite-dimensional set of all multifrequency solitary wave solutions with one, two, and four frequencies. The components of the solitary manifold corresponding to the solitary waves of the first two types are generically two-dimensional, while the component corresponding to the last type is generically four-dimensional. The attraction to the set of solitary waves is caused by the nonlinear energy transfer from lower harmonics to the continuous spectrum and subsequent radiation. For the proof, we develop the well-posedness for the nonlinear wave equation in discrete space-time, apply the technique of quasimeasures, and also obtain the version of the Titchmarsh convolution theorem for distributions on the circle.

To the memory of Boris Fedosov and Mark Vishik

1 Introduction

In this paper we study the long-time asymptotics for dispersive Hamiltonian systems. The first results in this direction were obtained by Segal [Seg63a, Seg63b], Strauss [Str68], and Morawetz and Strauss [MS72], who considered the nonlinear scattering and local convergence to zero for finite energy solutions to nonlinear wave equations. Apparently, there can be no such convergence to zero when there are localized standing wave solutions; in the case of $U(1)$ -invariant systems, these solutions are solitary waves of the form $\phi(x)e^{-i\omega t}$, with $\omega \in \mathbb{R}$ and ϕ decaying at infinity (one could say, “nonlinear Schrödinger eigenstates”). In this case, one expects that generically any finite energy solution breaks into a superposition of outgoing solitary waves and radiation; the statement known as the Soliton Resolution Conjecture (see [Sof06, Tao07]). The Soliton Resolution Conjecture implies that any finite energy solution locally converges either to zero or to a solitary wave. Thus, for a $U(1)$ -invariant dispersive Hamiltonian system, one expects that the weak attractor is formed by the set of all solitary waves. For a translation invariant system, this implies that the convergence to solitary waves is to take place – locally – in any inertial reference frame.

Existence of finite-dimensional attractors (formed by static stationary states) is extensively studied for *dissipative systems*, such as the Ginzburg-Landau, the Kuramoto-Sivashinsky, and the 2D forced Navier-Stokes equations, where the diffusive part of the equation damps higher frequencies and in some cases leads to existence of a finite-dimensional attractor [BV92, Tem97, CV02]. Existence of attractors for finite difference approximations of such dissipative systems, as well as the relation between the attractors of continuous systems and their approximations, was considered in [KK90, FT91, FJKT91].

We are interested in extending these results to the Hamiltonian systems, where the convergence to a certain attracting set (for both large positive and negative times) takes place not because of the dissipation, but instead due to the dispersion, and thus takes place “weakly”, in the weighted norms, with perturbations dispersing because of the local energy decay. In [KK07], we considered a weak attractor of the $U(1)$ -invariant nonlinear Klein-Gordon equation in one dimension, coupled to a nonlinear oscillator located at $x = 0$:

$$\partial_t^2 \psi(x, t) = \Delta \psi(x, t) - m^2 \psi(x, t) - \delta(x) p(|\psi(x, t)|^2) \psi(x, t), \quad x \in \mathbb{R}, \quad (1.1)$$

where $\psi(x, t) \in \mathbb{C}$ and $p(\cdot)$ is a potential with real coefficients, with positive coefficient at the leading order term. We proved in [KK07] that the attractor of all finite energy solutions is formed by the set of all solitary waves, $\phi_\omega(x)e^{-i\omega t}$, with $\omega \in \mathbb{R}$ and $\phi_\omega \in H^1(\mathbb{R})$. The general strategy of the proof has been to consider the omega-limit trajectories and then to prove that each omega-limit trajectory has a point spectrum, and thus is a solitary wave.

In this paper, we extend this result to the finite difference approximation of the n -dimensional Klein-Gordon equation interacting with a nonlinear oscillator. Our intention was to show that in the discrete case, just as in the continuous one, the attractor is formed by the set of solitary waves. This turned out to be true, except that in the discrete case, besides usual one-frequency solitary waves, the set of solitary wave solutions may contain the two- and four-frequency components. This is in agreement with our version of the Titchmarsh convolution theorem for distributions on the circle, which we needed to develop to complete the argument. These multifrequency solitary waves disappear in the continuous limit. To our knowledge, this is the first result on the weak attraction for the Hamiltonian model on discrete space-time.

The discretized models are widely studied in applied mathematics and in theoretical physics, in part due to atoms in a crystal forming a lattice, in part due to some of these models (such as the Ising model) being exactly solvable. Moreover, it is the discretized model that is used in numerical simulations of the continuous Klein-Gordon equation. The ground for considering the energy-conserving difference schemes for the nonlinear Klein-Gordon equations and nonlinear wave equations was set by Strauss and Vazquez in [SV78]. The importance of having conserved quantities in the numerical scheme was illustrated by noticing that instability occurs for the finite-difference schemes which do not conserve the energy [JV90]. Let us mention that our approach is also applicable to other energy-conserving finite-difference schemes so long as there are a priori bounds on the norm of the solution. Such schemes have been constructed in [LVQ95, Fur01, CJ10].

Our approach relies on the well-posedness results and the a priori estimates for the Strauss-Vazquez finite-difference scheme which we developed in [CK11]. While the discrete energy for the Strauss-Vazquez scheme given in [SV78] contained quadratic terms which in general are not positive-definite, we have shown [CK11] that the conserved discrete energy is positive-definite under the condition

$$\frac{\tau}{\varepsilon} \leq \frac{1}{\sqrt{n}} \quad (1.2)$$

on the grid ratio, where ε is the space step with respect to each component of $x \in \mathbb{R}^n$ and τ is the time step; the Strauss-Vazquez finite-difference scheme with the grid ratio $\tau/\varepsilon = 1/\sqrt{n}$ also preserves the discrete charge. The positive-definiteness of the conserved energy provides one with the a priori energy estimates and results in the stability of the finite-difference scheme. (The relation (1.2) agrees with the stability criterion in [Vir86].) While the charge conservation does not seem to be particularly important on its own, it could be considered as an indication that the $U(1)$ -invariance of the continuous equation is in a certain sense compatible with the chosen discretization procedure. See the discussion in [LVQ95, Section 1]. We reproduce our results on the well-posedness for the Strauss-Vazquez finite-difference scheme in Appendix A.

There is another important feature of our approach to the finite difference equation, compared to the approach which we developed in [KK07, KK08, KK10] for the continuous case. In the discrete case, *the spectral gap*, where the frequencies of the solitons are located and where, as it turns out, the spectrum of the omega-limit trajectory could be located, consists of *two* open neighborhoods of the circle. This does not allow us to apply the Titchmarsh convolution theorem in a direct form as in [KK07, KK08, KK10]. To circumvent this problem, we derive a version of the Titchmarsh convolution theorem for distributions on the circle; see Appendix B. This version of the Titchmarsh convolution theorem does not allow one to reduce the spectrum of omega-limit trajectories to a single point; we end up with the spectrum consisting of one, two, and four frequencies. Indeed such omega-limit trajectories exist; we explicitly construct solitary waves with one, two, and four frequencies.

Here is the plan of the paper. In Section 2, we describe the model and state the main results. In Section 3, we introduce the omega-limit trajectories and describe the proof of the main result: the convergence of any finite energy solution to the set of solitary waves. The main idea is that such a convergence is equivalent to showing that each omega-limit trajectory itself is a solitary wave. In Section 4, we separate the dispersive part of the solution, and consider the regularity of the remaining part in Section 5. In Section 6 we obtain the spectral relation satisfied by the omega-limit trajectory. For this, we use the technique of quasimeasures, which we borrow from [KK07]. In Section 7 we apply to the spectral relation our version of the Titchmarsh convolution theorem on the circle, proving that the spectrum of any omega-limit trajectory consists of finitely many frequencies. This completes the proof that each omega-limit trajectory is a (multifrequency) solitary wave. We give an explicit construction of multifrequency solitary waves in Section 8. Appendix A gives the well-posedness for the finite difference scheme approximation. The versions of the Titchmarsh convolution theorem for distributions on the circle are stated and proved in Appendix B.

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2 Definitions and main results

In [KK07], we considered the weak attractor of the $U(1)$ -invariant nonlinear Klein-Gordon equation in one dimension, coupled to a nonlinear oscillator located at $x = 0$:

$$\partial_t^2 \psi(x, t) = \Delta \psi(x, t) - m^2 \psi(x, t) - \delta(x) W'(|\psi(x, t)|^2) \psi(x, t), \quad x \in \mathbb{R}, \quad (2.1)$$

where $\psi(x, t) \in \mathbb{C}$ and $W(\cdot)$ is a real-valued polynomial which represents the potential energy of the oscillator:

$$W(|\psi|^2) = C_0 |\psi|^2 + C_1 |\psi|^4 + \dots + C_p |\psi|^{2(p+1)}, \quad p \geq 1, \quad C_p > 0.$$

Equation (2.1) is a Hamiltonian system with the Hamiltonian

$$\mathcal{E}(\psi, \dot{\psi}) = \frac{1}{2} \int_{\mathbb{R}} (|\dot{\psi}|^2 + |\psi'|^2 + m^2 |\psi|^2) dx + \frac{1}{2} W(|\psi(0, t)|^2). \quad (2.2)$$

In this paper, we will consider the discrete version of equation (2.1). We pick $\varepsilon > 0$ and $\tau > 0$ and substitute the continuous variables $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ by

$$x = \varepsilon X, \quad t = \tau T, \quad \text{where } (X, T) \in \mathbb{Z}^n \times \mathbb{Z}.$$

Remark 2.1. Note that we may couple a nonlinear oscillator to the Klein-Gordon field on the space-time lattice in any dimension $n \geq 1$. In the continuous case [KK07], one can only consider the dimension $n = 1$, when the Sobolev estimates (which we have due to the energy conservation) ensure that the solution is continuous as a function of x , so that the nonlinear term in (2.1) is well-defined.

From now on, we assume that $(X, T) \in \mathbb{Z}^n \times \mathbb{Z}$ is a point on the space-time integer lattice. Let $\psi \in l(\mathbb{Z}^n \times \mathbb{Z}, \mathbb{C})$ be a complex-valued function defined on this lattice. We will indicate dependence on the lattice points $\mathbb{Z}^n \times \mathbb{Z}$ by superscripts for the temporal dependence and by subscripts for the spatial dependence, so that ψ_X^T is the value of $\psi \in l(\mathbb{Z}^n \times \mathbb{Z}, \mathbb{C})$ at the point $X \in \mathbb{Z}^n$ at the moment $T \in \mathbb{Z}$.

The Strauss-Vazquez finite-difference scheme [SV78] applied to equation (2.1) takes the form

$$\frac{1}{\tau^2} D_T^2 \psi_X^T = \frac{1}{\varepsilon^2} D_X^2 \psi_X^T - m^2 \frac{\psi_X^{T+1} + \psi_X^{T-1}}{2} + \delta_{X,0} f^T, \quad X \in \mathbb{Z}^n, \quad T \in \mathbb{Z}, \quad (2.3)$$

where the nonlinear term is given by

$$f^T := \begin{cases} -\frac{W(|\psi_0^{T+1}|^2) - W(|\psi_0^{T-1}|^2)}{|\psi_0^{T+1}|^2 - |\psi_0^{T-1}|^2} \cdot \frac{\psi_0^{T+1} + \psi_0^{T-1}}{2} & \text{if } |\psi_0^{T+1}| \neq |\psi_0^{T-1}|; \\ -W'(|\psi_0^{T+1}|^2) \cdot \frac{\psi_0^{T+1} + \psi_0^{T-1}}{2} & \text{if } |\psi_0^{T+1}| = |\psi_0^{T-1}|. \end{cases} \quad (2.4)$$

In (2.3), we used the notations

$$D_T^2 \psi_X^T = \psi_X^{T+1} - 2\psi_X^T + \psi_X^{T-1}, \quad D_X^2 \psi_X^T = \sum_{j=1}^n (\psi_{X+e_j}^T - 2\psi_X^T + \psi_{X-e_j}^T), \quad (2.5)$$

with

$$e_1 = (1, 0, 0, 0, \dots) \in \mathbb{Z}^n, \quad e_2 = (0, 1, 0, 0, \dots) \in \mathbb{Z}^n, \quad \text{etc.} \quad (2.6)$$

Remark 2.2. By the little Bézout theorem,

$$\frac{W(|\psi_0^{T+1}|^2) - W(|\psi_0^{T-1}|^2)}{|\psi_0^{T+1}|^2 - |\psi_0^{T-1}|^2}$$

is a polynomial of $|\psi_0^{T\pm 1}|^2$. This polynomial coincides with the second line in (2.4) when $|\psi_0^{T-1}| = |\psi_0^{T+1}|$.

Assumption 2.3. $\frac{\tau}{\varepsilon} = \frac{1}{\sqrt{n}}$.

Assumption 2.4. $W(\lambda) = \sum_{q=0}^p C_q \lambda^{q+1}$, where $p \in \mathbb{N}$, $C_q \in \mathbb{R}$ for $0 \leq q \leq p$, and $C_p > 0$.

We introduce the phase space

$$\mathcal{X} = l^2(\mathbb{Z}^n) \times l^2(\mathbb{Z}^n), \quad \|(u, v)\|_{\mathcal{X}}^2 = \|u\|_{l^2}^2 + \|v\|_{l^2}^2, \quad (2.7)$$

where

$$\|u\|_{l^2}^2 = \sum_{X \in \mathbb{Z}^n} |u_X|^2, \quad u \in l^2(\mathbb{Z}^n).$$

We will denote by $l(\mathbb{Z}, \mathcal{X})$ the space of functions of $T \in \mathbb{Z}$ with values in \mathcal{X} .

Definition 2.5 (Discrete energy). The energy of the function $\psi \in l(\mathbb{Z}, \mathcal{X})$ at the moment $T \in \mathbb{Z}$ is

$$E^T = \sum_{X \in \mathbb{Z}^n} \varepsilon^n \left[\sum_{j=1}^n \frac{|\psi_X^{T+1} - \psi_{X-e_j}^T|^2 + |\psi_X^{T+1} - \psi_{X+e_j}^T|^2}{4n\tau^2} + m^2 \frac{|\psi_X^{T+1}|^2 + |\psi_X^T|^2}{4} \right] + \frac{W(|\psi_0^{T+1}|^2) + W(|\psi_0^T|^2)}{4}. \quad (2.8)$$

Remark 2.6. In the case $n = 1$ the continuous limit of the energy E in (2.8) coincides with the classical energy functional of the Klein-Gordon equation interacting with an oscillator described by the potential W ; see (2.2).

We consider the Cauchy problem

$$\begin{cases} D_T^2 \psi - \frac{1}{n} D_X^2 \psi + \tau^2 m^2 \frac{\psi_0^{T+1} + \psi_0^{T-1}}{2} = \tau^2 \delta_{X,0} f^T, & X \in \mathbb{Z}^n, \quad T \geq 1, \\ (\psi^T, \psi^{T+1})|_{T=0} = (u^0, u^1) \in \mathcal{X} = l^2(\mathbb{Z}^n) \times l^2(\mathbb{Z}^n), \end{cases} \quad (2.9)$$

where f^T is defined by (2.4).

Theorem 2.7 (Well-posedness). *Let $n \in \mathbb{N}$.*

- (i) *There is $\tau_0 > 0$ such that for any $\tau \in (0, \tau_0)$ and for all $(u^0, u^1) \in \mathcal{X}$ the Cauchy problem (2.9) has a unique solution $\psi \in l^\infty(\mathbb{Z}, l^2(\mathbb{Z}^n))$.*
- (ii) *The value of the energy functional is conserved: $E^T = E^0, T \in \mathbb{Z}$.*
- (iii) *ψ_X^T satisfies the a priori estimate*

$$\varepsilon^n \|\psi^T\|_{l^2}^2 \leq \frac{4}{m^2} [E^0 - \inf_{\lambda \geq 0} W(\lambda)], \quad T \in \mathbb{Z}. \quad (2.10)$$

- (iv) *For each $T \in \mathbb{Z}$, the map $U(T) : (u^0, u^1) \mapsto (\psi^T, \psi^{T+1})$ is continuous in \mathcal{X} .*

- (v) *For each $T \in \mathbb{Z}$, the map $U(T) : (u^0, u^1) \mapsto (\psi^T, \psi^{T+1})$ is weakly continuous in \mathcal{X} .*

The main part of this theorem (all the statements but the last one) is proved in [CK11]; we reproduce this proof in Appendix A (the existence, uniqueness, and continuous dependence on continuous data are proved in Theorems A.1, A.2, and A.4, and the a priori estimates are proved in Theorem A.8). Let us mention that the bound (2.10) follows from the energy conservation since the first term in (2.8) is nonnegative, while $\inf_{\lambda \geq 0} W(\lambda) > -\infty$ due to Assumption 2.4.

Let us sketch the proof of the weak continuity of $U(T)$, which is the last statement of the theorem. Let $\Psi_j \in \mathcal{X}$, $j \in \mathbb{N}$, be a sequence in \mathcal{X} weakly convergent to some $\Psi \in \mathcal{X}$. By the Banach-Steinhaus theorem, Ψ_j , $j \in \mathbb{N}$, are uniformly bounded in \mathcal{X} ; so are $U(T)(\Psi_j)$. Now one can see that the weak convergence of $U(T)(\Psi_j)$ to $U(T)(\Psi)$ in \mathcal{X} follows from the continuity of $U(T)$ in \mathcal{X} , from the finite speed of propagation (the value ψ_X^T only depends on the initial data (u_Y^0, u_Y^1) in the ball $|Y| \leq |X| + |T|$), and from the convergence $\Psi_j \rightarrow \Psi$ in the topology of $l^2(B)$ for any bounded set $B \subset \mathbb{Z}^n$.

We will use the standard notation

$$\mathbb{T} := \mathbb{R} \mod 2\pi.$$

Definition 2.8 (Solitary waves). (i) The solitary waves of equation (2.3) are solutions of the form

$$\psi^T = \sum_{k=1}^N \phi_k e^{-i\omega_k T}, \quad T \in \mathbb{Z}, \quad \text{where } \phi_k \in l^2(\mathbb{Z}^n), \quad \omega_k \in \mathbb{T}. \quad (2.11)$$

(ii) The solitary manifold is the set

$$\mathcal{S} = \{(\psi^T, \psi^{T+1})|_{T=0}\} \subset \mathcal{X} = l^2(\mathbb{Z}^n) \times l^2(\mathbb{Z}^n), \quad (2.12)$$

where ψ^T are solitary wave solutions to (2.3) of the form (2.11).

The set \mathcal{S} is nonempty since $\phi e^{-i\omega T}$ with $\phi \equiv 0$ is formally a solitary wave corresponding to any $\omega \in \mathbb{T}$.

Define

$$\omega_m := \arccos\left(1 + \frac{\tau^2 m^2}{2}\right)^{-1}. \quad (2.13)$$

Assumption 2.9. $\omega_m < \frac{\pi}{4(p+1)}$, where $p \in \mathbb{N}$ is defined in Assumption 2.4.

This assumption is needed so that the Titchmarsh convolution theorem for distributions on the circle (see Theorem 7.4 below) will be applicable for the analysis of omega-limit trajectories. One can see from (2.13) that for any fixed $m > 0$ Assumption 2.9 is satisfied as long as the time step $\tau > 0$ is sufficiently small.

Our main result is that the weak attractor of all finite energy solutions to (2.3) coincides with the solitary manifold \mathcal{S} .

Theorem 2.10 (Solitary manifold as the weak attractor). *Assume that $\tau \in (0, \tau_0)$, with $\tau_0 > 0$ as in Theorem 2.7. Let Assumptions 2.4 and 2.9 be satisfied. Then:*

- (i) *For any initial data $(u^0, u^1) \in \mathcal{X}$ the solution to the Cauchy problem (2.9) weakly converges to \mathcal{S} as $T \rightarrow \pm\infty$.*
- (ii) *The frequencies of the solitary waves (2.11) satisfy $\omega \in \Omega_0 \cup \Omega_\pi$ ($\omega_k \in \overline{\Omega_0} \cup \overline{\Omega_\pi}$ if $n \geq 5$), where the spectral gaps Ω_0 and Ω_π are defined by*

$$\Omega_0 = (-\omega_m, \omega_m) \subset \mathbb{T}, \quad \Omega_\pi = (\pi - \omega_m, \pi + \omega_m) \subset \mathbb{T}, \quad (2.14)$$

with $\omega_m = \arccos\left(1 + \frac{\tau^2 m^2}{2}\right)^{-1}$.

- (iii) *The set of all solitary wave solutions consists of solutions of the following three types:*

(a) *One-frequency solitary waves of the form*

$$\psi_X^T = \phi_X e^{-i\omega T}, \quad X \in \mathbb{Z}^n, \quad T \in \mathbb{Z}, \quad (2.15)$$

with $\phi \in l^2(\mathbb{Z}^n)$. The corresponding component of the solitary manifold is generically two-dimensional.

(b) *Two-frequency solitary waves of the form*

$$\psi_X^T = (1 + (-1)^{T+\Lambda \cdot X} \sigma) \phi_X e^{-i\omega T}, \quad X \in \mathbb{Z}^n, \quad T \in \mathbb{Z}, \quad (2.16)$$

with $\sigma \in \{\pm 1\}$, $\Lambda = (1, \dots, 1) \in \mathbb{Z}^n$, and $\phi \in l^2(\mathbb{Z}^n)$. The corresponding component of the solitary manifold is generically two-dimensional.

(c) *Four-frequency solitary waves of the form*

$$\psi_X^T = (1 + (-1)^{T+\Lambda \cdot X}) \phi_X e^{-i\omega T} + (1 - (-1)^{T+\Lambda \cdot X}) \theta_X e^{-i\omega' T}, \quad X \in \mathbb{Z}^n, \quad T \in \mathbb{Z}, \quad (2.17)$$

with $\Lambda = (1, \dots, 1) \in \mathbb{Z}^n$ and $\phi, \theta \in l^2(\mathbb{Z}^n)$. The corresponding component of the solitary manifold is generically four-dimensional.

Definition 2.8 and Theorem 2.10 show that the set \mathcal{S} satisfies the following two properties:

- (i) It is invariant under the evolution described by equation (2.3).
- (ii) It is the smallest set to which all finite energy solutions converge weakly.

It follows that \mathcal{S} is the weak attractor of all finite energy solutions to (2.3).

Remark 2.11. The convergence of any finite energy solution to \mathcal{S} , stated in Theorem 2.10, also holds in certain normed spaces. For example, let $s > 0$; for $u \in l^2(\mathbb{Z}^n)$, denote $\|u\|_{l^2_{-s}} = (\sum_{X \in \mathbb{Z}^n} |u_X|^2 (1 + X^2)^{-s})^{1/2}$, and for $(u, v) \in \mathcal{X}$, denote $\|(u, v)\|_{\mathcal{X}_{-s}} = (\|u\|_{l^2_{-s}}^2 + \|v\|_{l^2_{-s}}^2)^{1/2}$. Then, for any finite energy solution, one has

$$\text{dist}_{\mathcal{X}_{-s}}((\psi^T, \psi^{T+1}), \mathcal{S}) \xrightarrow{T \rightarrow \pm\infty} 0, \quad (2.18)$$

where $\text{dist}_{\mathcal{X}_{-s}}(\Psi, \mathcal{S}) = \inf_{\Phi \in \mathcal{S}} \text{dist}_{\mathcal{X}_{-s}}(\Psi, \Phi)$. This follows from the fact that the weak convergence in \mathcal{X} implies the strong convergence in \mathcal{X}_{-s} , for any $s > 0$.

3 Omega-limit trajectories

Here we explain our approach to the proof of Theorem 2.10. Since the equation is time-reversible, it suffices to prove the theorem for $T \rightarrow +\infty$. The following notion of omega-limit trajectory plays the crucial role in our approach.

Definition 3.1 (Omega-limit points and omega-limit trajectories).

- (i) $(z^0, z^1) \in \mathcal{X}$ is an omega-limit point of $\psi \in l(\mathbb{Z}, l^2(\mathbb{Z}^n))$ if there is a sequence $T_j \rightarrow +\infty$ such that $(\psi^{T_j}, \psi^{T_j+1})$ converges to (z^0, z^1) weakly in \mathcal{X} . We denote the set of all omega-limit points of $\psi \in l(\mathbb{Z}, l^2(\mathbb{Z}^n))$ by $\omega(\psi)$.
- (ii) An omega-limit trajectory of the solution ψ_X^T to (2.9) is a solution ζ_X^T to the discrete nonlinear Klein-Gordon equation (2.3),

$$D_T^2 \zeta_X^T - \frac{1}{n} D_X^2 \zeta_X^T + \tau^2 m^2 \frac{\zeta_X^{T+1} + \zeta_X^{T-1}}{2} = \tau^2 \delta_{X,0} g^T, \quad X \in \mathbb{Z}^n, \quad T \in \mathbb{Z}, \quad (3.1)$$

where (Cf. (2.4))

$$g^T = \begin{cases} -\frac{W(|\zeta_0^{T+1}|^2) - W(|\zeta_0^{T-1}|^2) \frac{\zeta_0^{T+1} + \zeta_0^{T-1}}{2}}{|\zeta_0^{T+1}|^2 - |\zeta_0^{T-1}|^2} & \text{if } |\zeta_0^{T+1}|^2 \neq |\zeta_0^{T-1}|^2, \\ -W'(|\zeta_0^{T+1}|^2) \frac{\zeta_0^{T+1} + \zeta_0^{T-1}}{2} & \text{if } |\zeta_0^{T+1}|^2 = |\zeta_0^{T-1}|^2, \end{cases} \quad T \in \mathbb{Z}, \quad (3.2)$$

with the initial data at an omega-limit point of ψ_X^T :

$$(\zeta^T, \zeta^{T+1})|_{T=0} = (z^0, z^1) \in \omega(\psi).$$

Lemma 3.2. If $(z^0, z^1) = \text{w-lim}_{T_j \rightarrow \infty} (\psi^{T_j}, \psi^{T_j+1})$ (weakly in \mathcal{X}) is an omega-limit point of $\psi \in l(\mathbb{Z}, l^2(\mathbb{Z}^n))$ and if $\zeta \in l(\mathbb{Z}, l^2(\mathbb{Z}^n))$ is the omega-limit trajectory with

$$(\zeta^T, \zeta^{T+1})|_{T=0} = (z^0, z^1),$$

then $\psi^{T_j+T} \rightarrow \zeta^T$, weakly in $l^2(\mathbb{Z}^n)$, and in particular there is the convergence

$$\zeta_X^T = \lim_{T_j \rightarrow \infty} \psi_X^{T_j+T}, \quad X \in \mathbb{Z}^n, \quad T \in \mathbb{Z}. \quad (3.3)$$

Proof. This immediately follows from the weak continuity of $U(T)$ stated in Theorem 2.7. \square

We will deduce Theorem 2.10 from the following proposition.

Proposition 3.3. Under the conditions of Theorem 2.10, any omega-limit trajectory ζ_X^T of some finite energy solution is a solitary wave of one of the following types:

- (i) $\zeta_X^T = \phi e^{-i\omega T}$, with $\phi \in l^2(\mathbb{Z}^n)$ and $\omega \in \Omega_0 \cup \Omega_\pi$ ($\omega \in \overline{\Omega_0} \cup \overline{\Omega_\pi}$ if $n \geq 5$);
- (ii) $\zeta_X^T = (1 + (-1)^{T+\Lambda \cdot X} \sigma) \phi_X e^{-i\omega T}$, with $\sigma \in \{\pm 1\}$, $\phi \in l^2(\mathbb{Z}^n)$, and $\omega \in \Omega_0$ ($\omega \in \overline{\Omega_0}$ if $n \geq 5$);
- (iii) $\zeta_X^T = (1 + (-1)^{T+\Lambda \cdot X}) \phi_X e^{-i\omega T} + (1 - (-1)^{T+\Lambda \cdot X}) \theta_X e^{-i\omega' T}$, with $\phi, \theta \in l^2(\mathbb{Z}^n)$ and $\omega, \omega' \in \Omega_0$ ($\omega, \omega' \in \overline{\Omega_0}$ if $n \geq 5$).

Proposition 3.3 can be used to complete the proof of Theorem 2.10, as follows.

Proof of Theorem 2.10. Let $T_j \in \mathbb{N}$, $j \in \mathbb{N}$, be a sequence such that $T_j \rightarrow +\infty$. By the Banach-Alaoglu theorem, a priori bounds on $\|(\psi^T, \psi^{T+1})\|_{\mathcal{X}}$ stated in Theorem 2.7 allow us to choose a subsequence $\{T_{j_r} : r \in \mathbb{N}\}$ such that

$$(\psi^{T_{j_r}}, \psi^{T_{j_r}+1}) \xrightarrow[r \rightarrow \infty]{} (z^0, z^1) \in \mathcal{X}, \quad \text{weakly in } \mathcal{X}. \quad (3.4)$$

Let $\zeta \in l(\mathbb{Z}, \mathcal{X})$ be the corresponding *omega-limit trajectory*, that is, the solution to the Cauchy problem (2.9) with the initial data $(\zeta^T, \zeta^{T+1})|_{T=0} = (z^0, z^1)$. By Proposition 3.3, ζ^T is a solitary wave. It follows that $(z^0, z^1) = (\zeta^T, \zeta^{T+1})|_{T=0} \in \mathcal{S}$. Thus, the first two statements of Theorem 2.10 follow from Proposition 3.3.

Let us prove the last statement of Theorem 2.10, namely, that the set of all solitary waves only consists of one-, two-, and four-frequency solitary waves. It will follow from Proposition 3.3 if we can show that each solitary wave solution is itself (its own) omega-limit trajectory, has to be of the form specified by Proposition 3.3.

Lemma 3.4. *Let $\psi^T = \sum_{k=1}^N \phi_k e^{-i\omega_k T}$, with $\phi_k \in l^2(\mathbb{Z}^n)$, $\omega_k \in \mathbb{T}$. Then*

$$(\psi^T, \psi^{T+1})|_{T=0} \in \omega(\psi),$$

so that ψ^T is the omega-limit trajectory of itself.

Proof. Pick any sequence $T_j \in \mathbb{N}$, $T_j \rightarrow \infty$ such that $\omega_1 T_j \rightarrow 0 \in \mathbb{T}$ as $j \rightarrow \infty$. Then either $\{\omega_2 T_j : j \in \mathbb{N}\}$ is dense in \mathbb{T} , or it is a subset of $\{\frac{k\pi}{q} \in \mathbb{T} : k \in \mathbb{Z}_{2q}\}$, for some $q \in \mathbb{N}$. In the former case, we take a subsequence T'_j of T_j such that $\omega_2 T'_j \rightarrow 0$ in \mathbb{T} ; In the latter case, we consider a new sequence, $T'_j = qT_j$, so that $\omega_2 T'_j = 0$ (and we still have $\omega_1 T'_j = q\omega_1 T_j \rightarrow 0 \in \mathbb{T}$).

Repeating this process, we end up with a sequence such that $\omega_k T_j \rightarrow 0 \in \mathbb{T}$ as $j \rightarrow \infty$ for all $1 \leq k \leq N$. It follows that $(\psi^{T_j}, \psi^{T_j+1}) \xrightarrow[j \rightarrow \infty]{\mathcal{X}} (\psi^0, \psi^1)$. By Definition 3.1, (ψ^0, ψ^1) is the omega-limit point of ψ^T ; that is, $(\psi^0, \psi^1) \in \omega(\psi)$.

Hence, ψ^T itself is an omega-limit trajectory of a finite energy solution, and has to be of one of the three types mentioned in Proposition 3.3. \square

The dimension of the components of the solitary manifold corresponding to one-, two-, and four-frequency solitary waves are computed in Lemma 8.1, Lemma 8.2, and Lemma 8.3 below.

This finishes the proof of Theorem 2.10. \square

It remains to prove Proposition 3.3, which is the contents of in the remaining part of the paper. We will prove it analyzing the spectrum of omega-limit trajectories. Everywhere below, we suppose that the conditions of Proposition 3.3 (that is, conditions of Theorem 2.10) hold.

4 Separation of the dispersive component

We rewrite (2.9) as a linear nonhomogeneous equation

$$(\mathbf{A}\psi)_X^T := D_T^2 \psi_X^T - \frac{1}{n} D_X^2 \psi_X^T + \tau^2 m^2 \frac{\psi_X^{T+1} + \psi_X^{T-1}}{2} = \tau^2 \delta_{X,0} f^T, \quad (X, T) \in \mathbb{Z}^n \times \mathbb{Z}, \quad (4.1)$$

$$(\psi^T, \psi^{T+1})|_{T=0} = (u^0, u^1) \in \mathcal{X} = (l^2(\mathbb{Z}^n) \times l^2(\mathbb{Z}^n)),$$

where f^T is given by (2.4).

Let $a(\xi, \omega)$ be the symbol of the operator \mathbf{A} in the left-hand side of (4.1):

$$a(\xi, \omega) := (2 + \tau^2 m^2) \cos \omega - \frac{2}{n} \sum_{j=1}^n \cos \xi_j, \quad \xi \in \mathbb{T}^n, \quad \omega \in \mathbb{T}. \quad (4.2)$$

For a fixed value of $\omega \in \mathbb{T}$, the dispersion relation

$$a(\xi, \omega) = 0, \quad \xi \in \mathbb{T}^n, \quad \omega \in \mathbb{T}, \quad (4.3)$$

admits a solution $\xi \in \mathbb{T}^n$ if and only if $|\cos \omega| \leq \left(1 + \frac{\tau^2 m^2}{2}\right)^{-1}$, or, equivalently, when ω belongs to the continuous spectrum Ω_c of the linear discrete equation (4.1), which is given by

$$\Omega_c = \mathbb{T} \setminus (\Omega_0 \cup \Omega_\pi); \quad (4.4)$$

the spectral gaps Ω_0 and Ω_π have been defined in (2.14).

Note that due to the factor $\frac{1}{n}$ in (4.1) the continuous spectrum does not depend on the dimension n .

Lemma 4.1. *If $n \leq 4$, the expression $\frac{1}{a(\xi, \omega)}$ is of finite L^2 -norm in $\xi \in \mathbb{T}^n$ if and only if $\omega \in \Omega_0 \cup \Omega_\pi$. If $n \geq 5$, $\frac{1}{a(\xi, \omega)}$ is of finite L^2 -norm in $\xi \in \mathbb{T}^n$ if and only if $\omega \in \overline{\Omega_0} \cup \overline{\Omega_\pi}$.*

Proof. When $\omega \in \Omega_c \setminus \partial\Omega_c$, $a(\xi, \omega)$ vanishes on the (singular) hypersurface in \mathbb{T}^n , then $\frac{1}{a(\xi, \omega)} \notin L^2(\mathbb{T}^n)$. If $\omega \in \Omega_0 \cup \Omega_\pi$, then $a(\xi, \omega)$ does not vanish for any $\xi \in \mathbb{T}^n$, hence $\frac{1}{a(\xi, \omega)}$ is of finite L^2 -norm in $\xi \in \mathbb{T}^n$.

For $\omega_b \in \partial\Omega_c = \{\pm\omega_m, \pi \pm \omega_m\}$, $a(\xi, \omega_b)$ vanishes at the points $\xi = (0, \dots, 0) \in \mathbb{T}^n$ and $\xi = (\pi, \dots, \pi) \in \mathbb{T}^n$. The consideration is the same in the neighborhoods of both of these points: near $\xi = (0, \dots, 0) \in \mathbb{T}^n$, one has $\frac{1}{a(\xi, \omega_b)} = \frac{1}{\frac{1}{n}\xi^2 + o(\xi^2)}$; therefore, for $n \leq 4$, one has $\frac{1}{a(\xi, \omega_b)} \notin L^2(\mathbb{T}^n)$; for $n \geq 5$, $\frac{1}{a(\xi, \omega_b)} \in L^2(\mathbb{T}^n)$. \square

For $T \geq 0$, we decompose the solution to (4.1) into $\psi_X^T = \chi_X^T + \varphi_X^T$, where χ, φ satisfy the equations

$$(A\chi)_X^T = 0 \quad X \in \mathbb{Z}^n, \quad T \geq 1; \quad (\chi^T, \chi^{T+1})|_{T=0} = (\psi^0, \psi^1), \quad (4.5)$$

$$(A\varphi)_X^T = \tau^2 \delta_{X,0} f^T, \quad X \in \mathbb{Z}^n, \quad T \geq 1; \quad (\varphi^T, \varphi^{T+1})|_{T=0} = (0, 0), \quad (4.6)$$

with f^T from (2.4). Note that both χ and φ are only defined for $T \geq 0$. Due to the energy conservation (Theorem 2.7), which certainly also takes place for the linear equation, the component χ is bounded in time:

$$\sup_{T \in \mathbb{Z}_+} \|\chi^T\|_{l^2(\mathbb{Z}^n)} < \infty. \quad (4.7)$$

Moreover, χ is purely dispersive in the following sense.

Lemma 4.2. *For any bounded $B \subset \mathbb{Z}^n$,*

$$\|\chi^T\|_{l^2(B)} \rightarrow 0, \quad |T| \rightarrow \infty. \quad (4.8)$$

Proof. The solution χ^T to (4.5) is given by

$$\chi_X^T = \int_{\mathbb{T}^n} \left(e^{i(\xi \cdot X + \omega(\xi)T)} \mathcal{P}_+(\xi) + e^{i(\xi \cdot X - \omega(\xi)T)} \mathcal{P}_-(\xi) \right) \frac{d\xi}{(2\pi)^n}, \quad (4.9)$$

where $\omega(\xi)$ is the unique solution to the dispersion relation (4.3) which satisfies $\omega(\xi) \in [\omega_m, \pi - \omega_m] \subset (0, \pi)$ (Cf. (4.4)), while functions $\mathcal{P}_\pm(\xi)$ are determined from the initial data (u^0, u^1) in (2.9) by

$$\hat{u}^0(\xi) = \mathcal{P}_+(\xi) + \mathcal{P}_-(\xi), \quad \hat{u}^1(\xi) = e^{i\omega(\xi)} \mathcal{P}_+(\xi) + e^{-i\omega(\xi)} \mathcal{P}_-(\xi).$$

Since $\det \begin{bmatrix} 1 & 1 \\ e^{i\omega(\xi)} & e^{-i\omega(\xi)} \end{bmatrix} = -2i \sin \omega(\xi)$, with $\inf_{\xi \in \mathbb{T}^n} |\sin \omega(\xi)| = \sin \omega_m > 0$, one can see that $\mathcal{P}_\pm \in L^2(\mathbb{T}^n)$, and moreover there is $C_{\tau, m} < \infty$ independent on u^0, u^1 such that

$$\|\mathcal{P}_\pm\|_{L^2(\mathbb{T}^n)} \leq C_{\tau, m} (\|u^0\|_{l^2(\mathbb{Z}^n)} + \|u^1\|_{l^2(\mathbb{Z}^n)}). \quad (4.10)$$

Further,

$$|\nabla_\xi(\xi \cdot X \pm \omega(\xi)T)| = |X \pm \nabla \omega(\xi)T|,$$

where $\nabla \omega(\xi)$ can be determined by differentiating the dispersion relation (4.3):

$$\nabla \omega(\xi) = \frac{1}{\left(1 + \frac{\tau^2 m^2}{2}\right) \sin \omega(\xi)} (\sin \xi_1, \dots, \sin \xi_n), \quad \xi \in \mathbb{T}^n,$$

where $\sin \omega(\xi) \geq \sin \omega_m > 0$. Thus, $\nabla \omega(\xi)$ vanishes only at the discrete set of points $\xi \in \{0; \pi\}^n \subset \mathbb{T}^n$. Hence, for any $\delta > 0$ we can choose a δ -neighborhood U_δ of the set $\{0; \pi\}^n \subset \mathbb{T}^n$ such that for any bounded $B \subset \mathbb{Z}^n$ there is $T_{\delta, B} > 0$ and $c_{\delta, B} > 0$ such that

$$|\nabla_\xi(\xi \cdot X \pm \omega(\xi)T)| \geq c_{\delta, B}|T|, \quad \xi \notin U_\delta, \quad X \in B, \quad |T| \geq T_{\delta, B}. \quad (4.11)$$

Let us fix a bounded set $B \subset \mathbb{Z}^n$.

Pick an arbitrary $\epsilon > 0$. We choose $\delta > 0$ sufficiently small and split the initial data \mathcal{P}_\pm into $\mathcal{P}_\pm(\xi) = \mathcal{R}_\pm(\xi) + \mathcal{S}_\pm(\xi)$, so that $\|\mathcal{R}_\pm\|_{L^2(\mathbb{T}^n)} < \epsilon/3$ while $\mathcal{S}_\pm(\xi)$ are smooth and supported outside a δ -neighborhood U_δ of $\{0; \pi\}^n \subset \mathbb{T}^n$. Substituting the splitting $\mathcal{P}_\pm = \mathcal{R}_\pm + \mathcal{S}_\pm$ into (4.9), we have $\chi_X^T = \rho_X^T + \sigma_X^T$, where

$$\rho_X^T = \int_{\mathbb{T}^n} \left(e^{i(\xi \cdot X + \omega(\xi)T)} \mathcal{R}_+(\xi) + e^{i(\xi \cdot X - \omega(\xi)T)} \mathcal{R}_-(\xi) \right) \frac{d\xi}{(2\pi)^n}, \quad (4.12)$$

$$\sigma_X^T = \int_{\mathbb{T}^n} \left(e^{i(\xi \cdot X + \omega(\xi)T)} \mathcal{S}_+(\xi) + e^{i(\xi \cdot X - \omega(\xi)T)} \mathcal{S}_-(\xi) \right) \frac{d\xi}{(2\pi)^n}. \quad (4.13)$$

Due to our choice of \mathcal{R}_\pm , one has

$$\|\rho^T\|_{l^2(\mathbb{Z}^n)} \leq \|\mathcal{R}_-\|_{L^2(\mathbb{T}^n)} + \|\mathcal{R}_+\|_{L^2(\mathbb{T}^n)} \leq 2\epsilon/3, \quad T \in \mathbb{Z}.$$

Using (4.11) to integrate by parts in (4.13), one proves that $\lim_{|T| \rightarrow \infty} \|\sigma^T\|_{l^2(B)} = 0$, so that $\|\sigma^T\|_{l^2(B)} < \epsilon/3$ for sufficiently large $|T|$. It follows that

$$\|\chi^T\|_{l^2(B)} \leq \|\rho^T\|_{l^2} + \|\sigma^T\|_{l^2(B)} < \epsilon$$

for sufficiently large $|T|$. Since $\epsilon > 0$ is arbitrary, we conclude that

$$\lim_{|T| \rightarrow \infty} \|\chi^T\|_{l^2(B)} = 0.$$

□

5 Regularity on the continuous spectrum

Now we consider the equation on φ^T ; see (4.6). Let us recall that f^T is defined by (2.4) for $T \in \mathbb{Z}$, but is only considered in (4.6) for $T \geq 1$. The function φ_X^T is defined by (4.6) for $T \geq 0$ (with $\varphi_X^0 = \varphi_X^1 = 0$). We extend f^T and φ^T by zeros for $T \leq 0$, so that

$$f^T = 0 \quad \text{for } T \leq 0, \quad \varphi^T = 0 \quad \text{for } T \leq 1. \quad (5.1)$$

Then equation (4.6) is satisfied for all $T \in \mathbb{Z}$:

$$(A\varphi)_X^T := D_T^2 \varphi_X^T - \frac{1}{n} D_X^2 \varphi_X^T + \tau^2 m^2 \frac{\varphi_X^{T+1} + \varphi_X^{T-1}}{2} = \tau^2 \delta_{X,0} f^T, \quad X \in \mathbb{Z}^n, \quad T \in \mathbb{Z}. \quad (5.2)$$

We introduce the Fourier transforms

$$\tilde{\varphi}_X(\omega) := \sum_{T \in \mathbb{N}} e^{i\omega T} \varphi_X^T, \quad X \in \mathbb{Z}^n, \quad \omega \in \mathbb{T}; \quad (5.3)$$

$$\hat{\varphi}(\xi, \omega) := \sum_{X \in \mathbb{Z}^n, T \in \mathbb{N}} e^{-i\xi \cdot X + i\omega T} \varphi_X^T, \quad \xi \in \mathbb{T}^n, \quad \omega \in \mathbb{T}; \quad (5.4)$$

$$\tilde{f}(\omega) := \sum_{T \in \mathbb{N}} e^{i\omega T} f^T, \quad \omega \in \mathbb{T}. \quad (5.5)$$

Since in the above relations the summation is over $T \in \mathbb{N}$, we can extend (5.3)–(5.5) to the upper half-plane as analytic functions of $\omega \in \mathbb{C}^+$. Then equation (4.6) yields

$$a(\xi, \omega) \hat{\varphi}(\xi, \omega) = \tau^2 \tilde{f}(\omega), \quad \xi \in \mathbb{T}^n, \quad \omega \in \mathbb{C}^+ \bmod 2\pi, \quad (5.6)$$

where $a(\xi, \omega)$ is defined by extending (4.2) to $\omega \in \mathbb{C}$:

$$a(\xi, \omega) := (2 + \tau^2 m^2) \cos \omega - \frac{2}{n} \sum_{j=1}^n \cos \xi_j, \quad \xi \in \mathbb{T}^n, \quad \omega \in \mathbb{C}. \quad (5.7)$$

Lemma 5.1. For $\xi \in \mathbb{T}^n$, $\omega \in \mathbb{C} \setminus \Omega_c$, one has $a(\xi, \omega) \neq 0$.

Proof. Recall that Ω_c was defined by (4.4) so that $a(\xi, \omega) \neq 0$ for $\xi \in \mathbb{T}^n$, $\omega \in \mathbb{T} \setminus \Omega_c$. For $\omega \in \mathbb{C} \setminus \mathbb{R}$, one has

$$a(\xi, \omega) = (2 + \tau^2 m^2) (\cos \operatorname{Re} \omega \cosh \operatorname{Im} \omega - i \sin \operatorname{Re} \omega \sinh \operatorname{Im} \omega) - \frac{2}{n} \sum_{j=1}^n \cos \xi_j.$$

Now it suffices to notice that if $\operatorname{Im} \omega \neq 0$ and $\operatorname{Re} \omega \notin \{0; \pi\}$, then $\operatorname{Im} a(\xi, \omega) \neq 0$, while for $\operatorname{Re} \omega \in \{0; \pi\}$ one has $|\operatorname{Re} a(\xi, \omega)| \geq \tau^2 m^2$. \square

By Lemma 5.1, for ω away from Ω_c , equation (5.6) yields

$$\hat{\varphi}(\xi, \omega) = \frac{\tau^2 \tilde{f}(\omega)}{a(\xi, \omega)}, \quad \xi \in \mathbb{T}^n, \quad \omega \in (\mathbb{C}^+ \bmod 2\pi) \setminus \Omega_c. \quad (5.8)$$

For $\omega \in (\mathbb{C}^+ \bmod 2\pi) \setminus \Omega_c$, since $\inf_{\xi \in \mathbb{T}^n} |a(\xi, \omega)| = \min_{\xi \in \mathbb{T}^n} |a(\xi, \omega)| > 0$, the operator of multiplication by $1/a(\xi, \omega)$ is a bounded linear operator from $L^2(\mathbb{T}^n)$ to $L^2(\mathbb{T}^n)$. In the coordinate representation, (5.8) can be written as

$$\tilde{\varphi}_X(\omega) = (\mathbf{R}(\omega) [\tau^2 \delta_{Y,0} \tilde{f}(\omega)])_X, \quad \omega \in (\mathbb{C}^+ \bmod 2\pi) \setminus \Omega_c, \quad X, Y \in \mathbb{Z}^n, \quad (5.9)$$

where $\mathbf{R}(\omega)$ is a bounded linear operator

$$\mathbf{R}(\omega) = \mathcal{F}^{-1} \circ \frac{1}{a(\xi, \omega)} \circ \mathcal{F} : l^2(\mathbb{Z}^n) \rightarrow l^2(\mathbb{Z}^n), \quad \omega \in (\mathbb{C}^+ \bmod 2\pi) \setminus \Omega_c, \quad (5.10)$$

with the Fourier transform $\mathcal{F} : l^2(\mathbb{Z}^n) \rightarrow L^2(\mathbb{T}^n)$ and its inverse given by

$$(\mathcal{F}u)(\xi) = \sum_{X \in \mathbb{Z}^n} e^{-i\xi \cdot X} u_X, \quad (\mathcal{F}^{-1}v)_X = \int_{\mathbb{T}^n} e^{i\xi \cdot X} v(\xi) \frac{d\xi}{(2\pi)^n}.$$

The expression in the right-hand side of (5.9) can be written as

$$(\mathbf{R}(\omega) [\tau^2 \delta_{Y,0} \tilde{f}(\omega)])_X = \sum_{Y \in \mathbb{Z}^n} \mathcal{G}_{X-Y}(\omega) \tau^2 \delta_{Y,0} \tilde{f}(\omega) = \mathcal{G}_X(\omega) \tau^2 \tilde{f}(\omega), \quad (5.11)$$

where $X \in \mathbb{Z}^n$, $\omega \in (\mathbb{C}^+ \bmod 2\pi) \setminus \Omega_c$, and $\mathcal{G}_X(\omega)$ stands for the fundamental solution, which is the inverse Fourier transform of $1/a(\xi, \omega)$:

$$\mathcal{G}_X(\omega) = \mathcal{F}^{-1} \left[\frac{1}{a(\xi, \omega)} \right] (X) = \int_{\mathbb{T}^n} \frac{e^{i\xi \cdot X}}{(2 + \tau^2 m^2) \cos \omega - \frac{2}{n} \sum_{j=1}^n \cos \xi_j} \frac{d\xi}{(2\pi)^n}, \quad (5.12)$$

where $\omega \in (\mathbb{C}^+ \bmod 2\pi) \setminus \Omega_c$.

Let us study properties of $\mathcal{G}_X(\omega)$. We start by introducing the set of the singular points in the continuous spectrum Ω_c :

$$\Sigma := \left\{ \pm \arccos \frac{1 - \frac{2l}{n}}{1 + \frac{\tau^2 m^2}{2}} : l = 0, 1, \dots, n \right\} \subset \Omega_c \subset \mathbb{T}. \quad (5.13)$$

These frequencies correspond to the critical points of the symbol $a(\xi, \omega)$, that is, for $\omega \in \Sigma$, there is $\xi \in \mathbb{T}^n$ such that both $a(\xi, \omega) = 0$ and $\nabla_{\xi} a(\xi, \omega) = 0$. (The relation $\nabla_{\xi} a(\xi, \omega) = 0$ implies that $\cos \xi_j = \pm 1$ for all $1 \leq j \leq n$; the value of l in (5.13) is the number of cosines in the denominator of (5.12) which are equal to -1 .) Note that in the one-dimensional case one has $\Sigma = \partial\Omega_c = \{\pm\omega_m; \pi \pm \omega_m\}$, with $\omega_m = \arccos(1 + \frac{\tau^2 m^2}{2})^{-1}$.

Lemma 5.2. (i) For each fixed value of $X \in \mathbb{Z}^n$, the function $\mathcal{G}_X(\omega)$ is analytic in $\omega \in (\mathbb{C}^+ \bmod 2\pi) \setminus \Omega_c$ and admits the trace

$$\mathcal{G}_X(\omega) := \mathcal{G}_X(\omega + i0), \quad \omega \in \mathbb{T} \setminus \Sigma, \quad (5.14)$$

which is a smooth function of $\omega \in \mathbb{T} \setminus \Sigma$.

(ii) For $X \in \mathbb{Z}^n$, $\omega \in \Omega_0 \cup \Omega_\pi$ (for $\omega \in \overline{\Omega_0} \cup \overline{\Omega_\pi}$ if $n \geq 5$), there are relations

$$\mathcal{G}_X(-\omega) = \mathcal{G}_X(\omega), \quad \mathcal{G}_X(\omega + \pi) = -(-1)^{\Lambda \cdot X} \mathcal{G}_X(\omega), \quad (5.15)$$

where $\Lambda = (1, \dots, 1) \in \mathbb{Z}^n$.

(iii) (a) If $n \leq 4$, $\mathcal{G}_0(\omega)$ is a monotonically increasing function on the interval $0 \leq \omega < \omega_m$, where it is strictly positive, and on the interval $\pi - \omega_m < \omega \leq \pi$ where it is strictly negative.

(b) If $n \geq 5$, $\mathcal{G}_0(\omega)$ is a monotonically increasing function on the interval $0 \leq \omega \leq \omega_m$, where it is strictly positive, and on the interval $\pi - \omega_m \leq \omega \leq \pi$, where it is strictly negative.

(iv) For every $X \in \mathbb{Z}^n$, the boundary trace $\mathcal{G}_X(\omega + i0)$ is a multiplier in $\mathcal{D}'(\mathbb{T} \setminus \Sigma)$, the space of distributions on $\mathbb{T} \setminus \Sigma$.

(v) For any closed set $I \subset \mathbb{T} \setminus (\overline{\Omega_0} \cup \overline{\Omega_\pi})$, there is $c_I > 0$ such that

$$\|\mathcal{G}(\omega + i\epsilon)\|_{l^2(\mathbb{Z}^n)}^2 \geq \frac{c_I}{\epsilon}, \quad \text{for all } \epsilon \in (0, 1), \quad \omega \in I. \quad (5.16)$$

Proof. 1. The analyticity follows directly from (5.12) and the definition (4.4) of Ω_c . The continuity of the traces for $\omega \in \mathbb{T} \setminus \Sigma$ follows by the Sokhotsky-Plemelj formula after taking the symbol $a(\xi, \omega)$ in (5.12) as a new coordinate function in a neighborhood of the hypersurface $\Gamma_\omega := \{\xi \in \mathbb{T}^n : a(\xi, \omega) = 0\}$; see e.g. [Esk67, SV01]. The smoothness follows similarly; see e.g. [Kop09, Proposition 2.2].

2. For $X \in \mathbb{Z}^n$, $\omega \in \Omega_0 \cup \Omega_\pi$, we have:

$$\begin{aligned} \mathcal{G}_X(\omega + \pi) &= \int_{\mathbb{T}^n} \frac{e^{i(\xi + \pi\Lambda) \cdot X}}{a(\xi + \pi\Lambda, \omega + \pi)} \frac{d\xi}{(2\pi)^n} \\ &= -(-1)^{\Lambda \cdot X} \int_{\mathbb{T}^n} \frac{e^{i\xi \cdot X}}{a(\xi, \omega)} \frac{d\xi}{(2\pi)^n} = -(-1)^{\Lambda \cdot X} \mathcal{G}_X(\omega). \end{aligned}$$

3. The monotonicity of $\mathcal{G}_0(\omega)$ for $\omega \in \mathbb{T} \setminus \Omega_c$ immediately follows from the definition (5.12). For $n \geq 5$, one notices that $\mathcal{G}_0(\omega)$ remains finite at $\omega = \pm\omega_m$ and $\omega = \pi \pm \omega_m$.

4. To prove that $\mathcal{G}_X(\omega + i0)$ is a multiplier in the space of distributions, it suffices to notice that for each $X \in \mathbb{Z}^n$ the trace $\mathcal{G}_X(\omega + i0)$ is a smooth function of $\omega \in \mathbb{T} \setminus \Sigma$.

5. Using (5.12) and the Plancherel theorem, we compute:

$$\begin{aligned} \|\mathcal{G}(\omega + i\epsilon)\|_{l^2(\mathbb{Z}^n)}^2 &= \sum_{X \in \mathbb{Z}^n} |\mathcal{G}_X(\omega + i\epsilon)|^2 = \int_{\mathbb{T}^n} \frac{1}{|a(\xi, \omega + i\epsilon)|^2} \frac{d\xi}{(2\pi)^n} \\ &\geq \int_{\mathbb{T}^n} \frac{1}{\left| (2 + \tau^2 m^2)(\cos \omega \cosh \epsilon - i \sin \omega \sinh \epsilon) - \frac{2}{n} \sum_{j=1}^n \cos \xi_j \right|^2} \frac{d\xi}{(2\pi)^n} \\ &\geq \int_{U_\epsilon(\Gamma_\omega)} \frac{1}{\left| (2 + \tau^2 m^2) \cos \omega \cosh \epsilon - \frac{2}{n} \sum_{j=1}^n \cos \xi_j \right|^2 + |(2 + \tau^2 m^2) \sin \omega \sinh \epsilon|^2} \frac{d\xi}{(2\pi)^n}, \end{aligned} \quad (5.17)$$

where $U_\epsilon(\Gamma_\omega) \subset \mathbb{T}^n$ is the ϵ -neighborhood of the level set $\Gamma_\omega \subset \mathbb{T}^n$ which is defined by the dispersion relation

$$\Gamma_\omega = \left\{ \xi \in \mathbb{T}^n : a(\xi, \omega) = 0 \right\}.$$

Note that since $\omega \in I \subset \mathbb{T} \setminus (\overline{\Omega_0} \cup \overline{\Omega_\pi})$, one has $(1 + \frac{\tau^2 m^2}{2}) |\cos \omega| < 1$, therefore Γ_ω is a nonempty submanifold of \mathbb{T}^n of codimension one, piecewise smooth away from the discrete set $\{0; \pi\}^n \subset \mathbb{T}^n$ (on this set, one could have simultaneously $a(\xi, \omega) = 0$, $\nabla_\xi a(\xi, \omega) = 0$; in fact, Γ_ω does not contain these points for $\omega \in \mathbb{T} \setminus \Sigma$). It follows that $|U_\epsilon(\Gamma_\omega)| = O(\epsilon)$. Moreover, since the hypersurface Γ_ω has strictly positive area for $\omega \in I$, there is $v_I > 0$ (dependent on I but not on ω) so that

$$|U_\epsilon(\Gamma_\omega)| \geq v_I \epsilon, \quad \forall \omega \in I.$$

One can see that for all $\xi \in U_\epsilon(\Gamma_\omega)$ and all $\epsilon \in (0, 1)$ the denominator in the integral in the right-hand side of (5.17) is bounded from above by $k_I \epsilon^2$, for some $k_I > 0$ which depends on I but not on $\omega \in I$. Thus, there is $c_I > 0$ such that

$$\|\mathcal{G}(\omega + i\epsilon)\|_{l^2(\mathbb{Z}^n)}^2 \geq \frac{c_I}{\epsilon}, \quad \epsilon \in (0, 1).$$

□

Lemma 5.3. $\tilde{f}|_{\mathbb{T} \setminus (\overline{\Omega_0} \cup \overline{\Omega_\pi})} \in L^2(\mathbb{T} \setminus (\overline{\Omega_0} \cup \overline{\Omega_\pi}))$.

Proof. We will prove this lemma considering $\tilde{f}(\omega)$ for $\omega \in \mathbb{C}^+$ and then taking the limit $\text{Im } \omega \rightarrow 0$. Since $f^T = 0$ for $T \leq 0$ (Cf. (5.1)) and f^T is bounded due to (2.10), the Fourier transform (5.5) extended to $\omega \in \mathbb{C}^+$,

$$\tilde{f}(\omega) = \sum_{T \in \mathbb{Z}} e^{i\omega T} f^T = \sum_{T \in \mathbb{N}} e^{i\omega T} f^T, \quad \omega \in \mathbb{C}^+ \bmod 2\pi, \quad (5.18)$$

defines an analytic function of $\omega \in \mathbb{C}^+$, which satisfies $|\tilde{f}(\omega)| \leq \frac{C}{1 - e^{-|\text{Im } \omega|}}$, $\omega \in \mathbb{C}^+$, with some $C < \infty$. Since for each $T \in \mathbb{N}$ there is a convergence

$$f^T e^{-\epsilon T} \rightarrow f^T, \quad \epsilon \rightarrow 0+,$$

\tilde{f} satisfies

$$\tilde{f}(\omega + i\epsilon) \rightarrow \tilde{f}(\omega), \quad \omega \in \mathbb{C}^+ \bmod 2\pi, \quad \epsilon \rightarrow 0+, \quad (5.19)$$

with the convergence in the sense of distributions. We need to show that for any closed subset $I \subset \mathbb{T} \setminus (\overline{\Omega_0} \cup \overline{\Omega_\pi})$ there is $C_I < \infty$ such that

$$\int_I |\tilde{f}(\omega + i\epsilon)|^2 d\omega < C_I, \quad \epsilon \in (0, 1). \quad (5.20)$$

If this were the case, then, taking into account the convergence (5.19), we would conclude that

$$\int_I |\tilde{f}(\omega)|^2 d\omega \leq C_I$$

by the Banach-Alaoglu theorem on weak compactness, finishing the proof.

It remains to prove (5.20). Similarly to (5.18), we define

$$\tilde{\varphi}_X(\omega) = \sum_{T \in \mathbb{N}} e^{i\omega T} \varphi_X^T, \quad X \in \mathbb{Z}^n, \quad \omega \in \mathbb{C}^+ \bmod 2\pi, \quad (5.21)$$

which is an analytic function valued in $l^2(\mathbb{Z}^n)$. Its limit as $\text{Im } \omega \rightarrow 0+$ exists as an element of $\mathcal{D}'(\mathbb{T}, l^2(\mathbb{Z}^n))$. Due to equation (5.2), the complex Fourier transforms of φ_X^T and of f^T are related by

$$\tilde{\varphi}_X(\omega) = \tau^2 \mathcal{G}_X(\omega) \tilde{f}(\omega), \quad \omega \in \mathbb{C}^+ \bmod 2\pi. \quad (5.22)$$

Using (5.22) and the Plancherel theorem, we see that

$$\tau^4 \int_{\mathbb{T}} \|\mathcal{G}(\omega + i\epsilon)\|_{l^2(\mathbb{Z}^n)}^2 |\tilde{f}(\omega + i\epsilon)|^2 \frac{d\omega}{2\pi} = \sum_{T \in \mathbb{N}} \|\varphi^T\|_{l^2(\mathbb{Z}^n)}^2 e^{-2\epsilon T} \leq \frac{\sup_{T \in \mathbb{Z}_+} \|\varphi^T\|_{l^2(\mathbb{Z}^n)}^2}{1 - e^{-2\epsilon}}, \quad (5.23)$$

where

$$\sup_{T \in \mathbb{Z}_+} \|\varphi^T\|_{l^2(\mathbb{Z}^n)} < \infty \quad (5.24)$$

due to the a priori estimates on ψ and χ (see (2.10) and (4.7)).

Combining the bound (5.23) with the bound on $\|\mathcal{G}(\omega + i\epsilon)\|_{l^2(\mathbb{Z}^n)}$ obtained in Lemma 5.2 (Cf. (5.16)), we conclude that there is $C_I < \infty$ such that (5.20) holds. This completes the proof of the lemma. \square

6 Spectral relation

Let $\zeta \in l(\mathbb{Z}, l^2(\mathbb{Z}^n))$ be the omega-limit trajectory of the solution $\psi \in l(\mathbb{Z}, l^2(\mathbb{Z}^n))$ to the Cauchy problem (2.9), in the sense of Definition 3.1. That is, we assume that ζ is a solution to (3.1) and that there is a sequence $T_j \in \mathbb{N}$, $j \in \mathbb{N}$, such that $(\psi^{T_j}, \psi^{T_j+1})$ converge weakly to $(\zeta^T, \zeta^{T+1})|_{T=0}$.

Let us express $\tilde{\zeta}_X(\omega)$ in terms of $\tilde{g}(\omega)$; this representation will allow us to express ζ_X^T in equation (3.1) via ζ_0^T . By the definition of omega-limit trajectory $\tilde{\zeta}_X(\omega)$ (see Definition 3.1), its Fourier transform in T satisfies the stationary Helmholtz-type equation

$$\left((2 + \tau^2 m^2) \cos \omega - 2 - \frac{1}{n} D_X^2 \right) \tilde{\zeta}_X(\omega) = \tau^2 \delta_{X,0} \tilde{g}(\omega), \quad X \in \mathbb{Z}^n, \quad \omega \in \mathbb{T}. \quad (6.1)$$

Lemma 6.1. *The solution to the stationary problem (6.1) satisfies*

$$\tilde{\zeta}_X(\omega) = \tau^2 \mathcal{G}_X(\omega) \tilde{g}(\omega), \quad \omega \in \mathbb{T} \setminus \Sigma, \quad X \in \mathbb{Z}^n, \quad (6.2)$$

where Σ is the set of singular points defined in (5.13).

Proof. By (6.1), the Fourier transform of ζ in X and T ,

$$\hat{\zeta}(\xi, \omega) = \sum_{X \in \mathbb{Z}^n, T \in \mathbb{Z}} \zeta_X^T e^{-i\xi \cdot X} e^{i\omega T}, \quad \xi \in \mathbb{T}^n, \quad \omega \in \mathbb{T},$$

satisfies

$$\left((2 + \tau^2 m^2) \cos \omega - \frac{2}{n} \sum_{j=1}^n \cos \xi_j \right) \hat{\zeta}(\xi, \omega) = a(\xi, \omega) \hat{\zeta}(\xi, \omega) = \tau^2 \tilde{g}(\omega), \quad (6.3)$$

where $\xi \in \mathbb{T}^n$ and $\omega \in \mathbb{T}$. By (2.4), (3.2), and (3.3),

$$f_j^T := f^{T+T_j} \xrightarrow{j \rightarrow \infty} g^T, \quad \forall T \in \mathbb{Z}, \quad (6.4)$$

and moreover the functions $\{f_j\}$ are uniformly bounded in $l^\infty(\mathbb{Z})$ due to (2.10). This is enough to conclude that f_j converge to g in the sense of tempered distributions on \mathbb{Z} (dual to $\mathcal{S}'(\mathbb{Z})$, which is the vector space of sequences decaying faster than any power of n), hence, due to continuity of the Fourier transform in the space of tempered distributions and due to $\mathcal{S}'(\mathbb{T}) = \mathcal{D}'(\mathbb{T})$,

$$\tilde{f}(\omega) e^{-i\omega T_j} \xrightarrow{j \rightarrow \infty} \tilde{g}(\omega). \quad (6.5)$$

Due to (3.3) and Lemma 4.2,

$$\zeta_X^T = \lim_{T_j \rightarrow \infty} \psi_X^{T_j+T} = \lim_{T_j \rightarrow \infty} \varphi_X^{T_j+T}, \quad X \in \mathbb{Z}^n, \quad T \in \mathbb{Z}. \quad (6.6)$$

Moreover, by Theorem 2.7, the solutions ψ_X^T, χ_X^T (Cf. (4.5)), and hence their difference $\varphi_X^T = \psi_X^T - \chi_X^T$ are bounded uniformly in T and X . Therefore, similarly to how we arrived at (6.5),

$$\tilde{\varphi}_X(\omega) e^{-i\omega T_j} \xrightarrow{j \rightarrow \infty} \tilde{\zeta}_X(\omega). \quad (6.7)$$

Now the proof follows from taking the limit $\text{Im } \omega \rightarrow 0+$ in (5.22) and using (6.5) and (6.7), and also taking into account that for each $X \in \mathbb{Z}^n$, $\mathcal{G}_X(\omega)$ is a multiplier in $\mathcal{D}'(\mathbb{T} \setminus \Sigma)$ by Lemma 5.2. \square

Let us show that the spectra of ζ_0^T and g^T are located inside the closures of the spectral gaps.

Lemma 6.2. $\text{supp } \tilde{g} \subset \overline{\Omega_0} \cup \overline{\Omega_\pi}$.

Proof. By Lemma 5.3, for any $\varrho \in C_0^\infty(\mathbb{T})$ with $\text{supp } \varrho \in \mathbb{T} \setminus (\overline{\Omega_0} \cup \overline{\Omega_\pi})$, one has $\varrho(\omega) \tilde{f}(\omega) \in L^1(\mathbb{T})$. Hence, by the Riemann-Lebesgue lemma,

$$\varrho(\omega) \tilde{f}(\omega) e^{-i\omega T_j} \xrightarrow{j \rightarrow \infty} 0$$

in the sense of distributions on \mathbb{T} . Due to (6.5), we conclude that $\tilde{g} = 0$ on $\mathbb{T} \setminus (\overline{\Omega_0} \cup \overline{\Omega_\pi})$. \square

Lemma 6.3. *A point $\omega \in \Omega_c$ can not be an isolated point of the support of $\tilde{\zeta}_0$.*

Proof. Assume that $\omega_1 \in \Omega_c \setminus \partial\Omega_c$ is an isolated point of the support of $\tilde{\zeta}_0$. By Lemma 6.2 we know that $\omega_1 \notin \text{supp } \tilde{g}$. Then there is $\varrho \in C_0^\infty(\mathbb{T})$ such that $\varrho(\omega_1) = 1$ and $\text{supp } \varrho \cap \text{supp } \tilde{g} = \emptyset$. Due to the spectral representation (6.2), we see that for any $X \in \mathbb{Z}^n$ one has $\text{supp } \varrho \cap \text{supp } \tilde{\zeta}_X \subset \{\omega_1\}$. Therefore, $\varrho(\omega) \tilde{\zeta}_X(\omega) = \delta(\omega - \omega_1) M_X$, where $M \in l^2(\mathbb{Z}^n)$ and $M \neq 0$. (The terms of the form $N_X \delta^{(k)}(\omega - \omega_1)$, $N \in l^2(\mathbb{Z}^n)$, with $k \geq 1$ do not appear due to the a priori bounds on ψ_X^T ; see Theorem 2.7.) The relation (6.1) implies that M satisfies the equation

$$-2(1 - \cos \omega_1) M_X - \frac{1}{n} D_X^2 M_X + \tau^2 m^2 M_X \cos \omega_1 = 0, \quad (6.8)$$

hence its Fourier transform satisfies

$$\left[(2 + \tau^2 m^2) \cos \omega_1 - \frac{2}{n} \sum_{j=1}^n \cos \xi_j \right] \hat{M}(\xi) = 0. \quad (6.9)$$

It follows that \hat{M} is supported on the hypersurface $\Gamma_{\omega_1} = \{\xi \in \mathbb{T}^n : a(\xi, \omega_1) = 0\} \subset \mathbb{T}^n$. (This hypersurface has singular points if $\omega_1 \in \Sigma$; see (5.13).) Since there is no nonzero $\hat{M} \in l^2(\mathbb{T}^n)$ supported on such a hypersurface, we arrive at a contradiction; hence, $\omega_1 \in \Omega_c \setminus \partial\Omega_c$ can not be an isolated point of the support of $\tilde{\zeta}_0$. \square

Lemma 6.4. $\text{supp } \tilde{\zeta}_0 \subset \overline{\Omega_0} \cup \overline{\Omega_\pi}$.

Proof. This inclusion follows from Lemma 6.2, the spectral representation (6.2), and Lemma 6.3. \square

We will use the construction of quasimeasures [KK07]. Denote by \check{q} the inverse Fourier transform of $q \in \mathcal{D}'(\mathbb{T})$:

$$\check{q}(T) = \mathcal{F}^{-1}[q(\omega)](T) = \int_{\mathbb{T}} e^{-i\omega T} q(\omega) \frac{d\omega}{2\pi}, \quad T \in \mathbb{Z}.$$

Definition 6.5 (Quasimeasures). (i) The space $l_F^\infty(\mathbb{Z})$ is the vector space $l^\infty(\mathbb{Z})$ endowed with the following convergence:

$$f_\epsilon(T) \xrightarrow[\epsilon \rightarrow 0+]{l_F^\infty} f(T)$$

if

$$\sup_{\epsilon > 0} \|f_\epsilon\|_{l^\infty(\mathbb{Z})} < \infty \quad \text{and} \quad \forall T_1 \in \mathbb{N}, \quad \lim_{\epsilon \rightarrow 0+} \sup_{T \in \mathbb{Z}, |T| \leq T_1} |f_\epsilon(T) - f(T)| \rightarrow 0.$$

(ii) The space of *quasimeasures* is the vector space of distributions with bounded Fourier transform,

$$\mathcal{Q}(\mathbb{T}) = \{q \in \mathcal{D}'(\mathbb{T}) : \check{q} \in l^\infty(\mathbb{Z})\},$$

endowed with the following convergence:

$$q_\epsilon(\omega) \xrightarrow[\epsilon \rightarrow 0+]{\mathcal{Q}} q(\omega) \quad \text{if} \quad \check{q}_\epsilon(T) \xrightarrow[\epsilon \rightarrow 0+]{l_F^\infty} \check{q}(T).$$

For example, any function from $L^1(\mathbb{T})$ is a quasimeasure, and so is any finite Borel measure on \mathbb{T} .

Let $M \in C(\mathbb{T})$, and let $\mathbf{M} : C(\mathbb{T}) \rightarrow C(\mathbb{T})$ be the operator of multiplication by M :

$$\text{for } u \in C(\mathbb{T}), \quad \mathbf{M} : u \mapsto Mu \in C(\mathbb{T}).$$

Lemma 6.6 (Multipliers in the space of quasimeasures).

(i) If $M \in C(\mathbb{T})$ is such that $\check{M} \in l^1(\mathbb{Z})$, then

$$\mathbf{M} : C(\mathbb{T}) \rightarrow C(\mathbb{T}), \quad \mathbf{M} : u \mapsto Mu$$

extends to a bounded linear operator $\mathbf{M} : \mathcal{Q}(\mathbb{T}) \rightarrow \mathcal{Q}(\mathbb{T})$.

(ii) Let $\check{M}_\epsilon \in l^1(\mathbb{Z})$ be bounded uniformly for $\epsilon > 0$. If

$$q_\epsilon \xrightarrow[\epsilon \rightarrow 0+]{\mathcal{Q}(\mathbb{T})} q, \quad \check{M}_\epsilon \xrightarrow[\epsilon \rightarrow 0+]{l^1(\mathbb{Z})} \check{M}, \quad (6.10)$$

$$\text{then } M_\epsilon q_\epsilon \xrightarrow[\epsilon \rightarrow 0+]{\mathcal{Q}(\mathbb{T})} Mq.$$

Proof. We define $M(\omega)q(\omega) := \mathcal{F}[(\check{M} * \check{q})(T)](\omega)$ that agrees with the case $q \in C(\mathbb{T})$. The statement (1) follows from (2) with $M_\epsilon = M$ and $q_\epsilon \in C(\mathbb{T})$. To prove (2), by Definition 6.5, we need to show that

$$\mathcal{F}^{-1}[M_\epsilon(\omega)q_\epsilon(\omega)](T) = (\check{M}_\epsilon * \check{q}_\epsilon)(T) \xrightarrow[\epsilon \rightarrow 0+]{l_F^\infty} (\check{M} * \check{q})(T). \quad (6.11)$$

Define the functions

$$f_\epsilon(T) := \mathcal{F}^{-1}[M_\epsilon(\omega)q_\epsilon(\omega)](T) = (\check{M}_\epsilon * \check{q}_\epsilon)(T),$$

$$f(T) := \mathcal{F}^{-1}[M(\omega)q(\omega)](T) = (\check{M} * \check{q})(T).$$

By Definition 6.5, to prove the convergence (6.11), we need to show that

$$\sup_{\epsilon > 0} \|f_\epsilon\|_{l^\infty(\mathbb{Z})} < \infty \quad (6.12)$$

and that for any $T_1 \in \mathbb{N}$ one has

$$\lim_{\epsilon \rightarrow 0+} \sup_{|T| < T_1} |f_\epsilon(T) - f(T)| = 0. \quad (6.13)$$

To prove (6.12), we write:

$$\|f_\epsilon\|_{l^\infty(\mathbb{Z})} \leq \sum_{T' \in \mathbb{Z}} |\check{M}_\epsilon(T')\check{q}_\epsilon(T - T')| \leq \|\check{M}_\epsilon\|_{l^1(\mathbb{Z})} \|\check{q}_\epsilon\|_{l^\infty(\mathbb{Z})},$$

which is bounded uniformly in $\epsilon > 0$.

It remains to prove (6.13). We need to show that, given $T_1 \in \mathbb{N}$, for any $\delta > 0$ there is $\epsilon_\delta > 0$ such that for any $\epsilon \in (0, \epsilon_\delta)$ one has $\sup_T |f_\epsilon(T) - f(T)| < \delta$. We have:

$$f_\epsilon(T) - f(T) = (\check{M}_\epsilon * \check{q}_\epsilon)(T) - (\check{M} * \check{q})(T) = ((\check{M}_\epsilon - \check{M}) * \check{q}_\epsilon)(T) + (\check{M} * (\check{q}_\epsilon - \check{q}))(T). \quad (6.14)$$

The first term in the right-hand side of (6.14) converges to zero uniformly in $T \in \mathbb{Z}$ since $\check{M}_\epsilon - \check{M} \rightarrow 0$ in $l^1(\mathbb{Z})$ while $\check{q}_\epsilon \in l^\infty(\mathbb{Z})$ are bounded uniformly for $\epsilon > 0$. If $\epsilon_\delta > 0$ is small enough and $\epsilon \in (0, \epsilon_\delta)$, then the first term in the right-hand side of (6.14) is smaller than $\delta/3$. We break the second term in the right-hand side of (6.14) into

$$\sum_{|T'| > T_\delta} \check{M}(T')(\check{q}_\epsilon(T - T') - \check{q}(T - T')) + \sum_{|T'| \leq T_\delta} \check{M}(T')(\check{q}_\epsilon(T - T') - \check{q}(T - T')), \quad (6.15)$$

where $T_\delta \in \mathbb{N}$ is chosen as follows: Since $\check{M} \in l^1(\mathbb{Z})$, while $\check{q}_\epsilon - \check{q}$ is bounded in $l^\infty(\mathbb{Z})$ uniformly in $\epsilon > 0$, there exists $T_\delta \in \mathbb{N}$ so that

$$\sum_{T' \in \mathbb{Z}, |T'| > T_\delta} |\check{M}(T')| |\check{q}_\epsilon(T - T') - \check{q}(T - T')| < \delta/3. \quad (6.16)$$

On the other hand, since $q_\epsilon \rightarrow q$ in $\mathcal{Q}(\mathbb{T})$, one has

$$\lim_{\epsilon \rightarrow 0} \sup_{|T| \leq T_1} \sup_{|T'| \leq T_\delta} |\check{q}_\epsilon(T - T') - \check{q}(T - T')| \leq \lim_{\epsilon \rightarrow 0} \sup_{|T| \leq T_1 + T_\delta} |\check{q}_\epsilon(T) - \check{q}(T)| = 0, \quad (6.17)$$

so that, choosing $\epsilon_\delta > 0$ smaller than necessary, we make sure that the second term in (6.15) is also smaller than $\delta/3$ for $\epsilon \in (0, \epsilon_\delta)$, and therefore (6.14) is bounded by δ .

Thus, as $\epsilon \rightarrow 0+$, (6.14) converges to zero uniformly in $|T| \leq T_1$, proving (6.13). The convergence (6.11) follows. \square

Denote $\Sigma' = \Sigma \setminus \partial\Omega_c$, where $\partial\Omega_c = \{\pm\omega_m, \pi \pm \omega_m\}$, with $\omega_m = \arccos(1 + \frac{\tau^2 m^2}{2})^{-1}$.

Lemma 6.7. For $n \geq 1$, the function

$$r(\omega) = \frac{1}{\mathcal{G}_0(\omega + i0)}, \quad \omega \in \mathbb{T}, \quad (6.18)$$

is continuous and real-valued for $\omega \in \overline{\Omega_0} \cup \overline{\Omega_\pi}$ and satisfies

$$r(\omega) = -r(\pi + \omega), \quad r(\omega) = r(-\omega), \quad \omega \in \overline{\Omega_0} \cup \overline{\Omega_\pi}; \quad r|_{\Omega_0} > 0, \quad r|_{\Omega_\pi} < 0. \quad (6.19)$$

It is a multiplier in the space of quasimeasures $\mathcal{Q}(\mathbb{T} \setminus \Sigma')$, and moreover for any $\rho \in C^\infty(\mathbb{T})$ with support away from Σ' one has

$$\mathcal{F}^{-1}\left[\frac{\rho(\omega)}{\mathcal{G}_0(\omega + i\epsilon)}\right](T) \xrightarrow[\epsilon \rightarrow 0+]{l^1(\mathbb{Z})} \mathcal{F}^{-1}[\rho(\omega)r(\omega)](T). \quad (6.20)$$

Remark 6.8. Due to Lemma 6.6, one concludes that Lemma 6.7 implies that, as $\omega \rightarrow 0+$, the ratio $\frac{1}{\mathcal{G}_0(\omega+i\epsilon)}$ converges to $r(\omega)$ in the space of multipliers which act on quasimeasures with support in $\mathbb{T} \setminus \Sigma'$.

Proof. The relations (6.19) follow from (5.12).

Since $\mathcal{G}_0(\omega)$ is a smooth function of $\omega \in (\mathbb{C}^+ \bmod 2\pi) \setminus \Sigma'$, it is enough to check the convergence (6.20) for the Fourier transform

$$\mathcal{F}^{-1} \left[\frac{\rho(\omega)}{\mathcal{G}_0(\omega+i\epsilon)} \right] (T), \quad T \in \mathbb{Z}, \quad \epsilon \in (0, 1), \quad (6.21)$$

with $\rho \in C_0^\infty(\mathbb{T})$ supported in a small open neighborhood of $\Sigma \setminus \Sigma' = \partial\Omega_c = \{\pm\omega_m, \pi \pm \omega_m\} \subset \mathbb{T}$, such that $\Sigma' \cap \text{supp } \rho = \emptyset$. We have:

$$\mathcal{G}_0(\omega+i\epsilon) = \frac{1}{2} \int_{\mathbb{T}^n} \frac{1}{\left(1 + \frac{\tau^2 m^2}{2}\right) \cos(\omega+i\epsilon) - \frac{1}{n} \sum_{j=1}^n \cos \xi_j} \frac{d\xi}{(2\pi)^n}.$$

In the case $n \geq 3$, the convergence of (6.21) in l^1 as $\epsilon \rightarrow 0$ is straightforward since at the points $\omega \in \partial\Omega_c$ the function $\mathcal{G}_0(\omega+i\epsilon)$ has a nonzero limit as $\epsilon \rightarrow 0$.

We leave the case $n = 2$ to the reader and consider the case $n = 1$. It suffices to consider the case when ρ is supported in a small neighborhood of $\omega = \omega_m$ (all other cases $\omega \in \partial\Omega_c$ are handled similarly). Let $\epsilon \in (0, 1)$. One evaluates $\mathcal{G}_0(\omega+i\epsilon)$ explicitly, getting

$$\begin{aligned} \mathcal{G}_0(\omega+i\epsilon) &= \frac{1}{2} \int_{\mathbb{T}} \frac{1}{\left(1 + \frac{\tau^2 m^2}{2}\right) \cos(\omega+i\epsilon) - \cos \xi} \frac{d\xi}{2\pi} \\ &= \frac{1}{2} \frac{1}{\sqrt{\left(1 + \frac{\tau^2 m^2}{2}\right)^2 \cos^2(\omega+i\epsilon) - 1}}, \end{aligned}$$

hence

$$\begin{aligned} \frac{\rho(\omega)}{\mathcal{G}_0(\omega+i\epsilon)} &= (2 + \tau^2 m^2) \rho(\omega) \sqrt{\cos^2(\omega+i\epsilon) - \cos^2 \omega_m} \\ &= (2 + \tau^2 m^2) \rho(\omega) \sqrt{\sin^2 \omega_m - \sin^2(\omega+i\epsilon)}, \end{aligned}$$

which can be written as

$$\rho(\omega) f(\omega, \epsilon) \sqrt{\omega - \omega_m + i\epsilon},$$

with $f(\omega, \epsilon)$ begin smooth on the support of ρ , with the two derivatives bounded uniformly for $\epsilon \in (0, 1)$.

It suffices to show that

$$F(T, \epsilon) = \int_{\mathbb{T}} e^{-i\omega T} \rho(\omega) f(\omega, \epsilon) \sqrt{\omega - \omega_m + i\epsilon} d\omega, \quad T \in \mathbb{Z}, \quad \epsilon \in (0, 1),$$

decays as $|T|^{-3/2}$, uniformly in $\epsilon \in (0, 1)$.

We pick $\alpha \in C^\infty(\mathbb{R})$, $\alpha|_{|s| \geq 2} \equiv 1$, $\alpha|_{|s| \leq 1} \equiv 0$, and define $\beta(s) = \alpha(s) - \alpha(s/2)$, so that $\beta \in C^\infty([1, 4])$. Then there is the dyadic decomposition

$$1 = \alpha(s) + \sum_{k=1}^{\infty} \beta(2^k s), \quad s \in \mathbb{R}.$$

For $T \in \mathbb{Z}$, $\epsilon \in (0, 1)$, we define

$$F_0(T, \epsilon) = \int_{\mathbb{T}} \alpha(|\omega - \omega_m + i\epsilon|) e^{-i\omega T} \rho(\omega) f(\omega, \epsilon) \sqrt{\omega - \omega_m + i\epsilon} d\omega, \quad (6.22)$$

$$F_k(T, \epsilon) = \int_{\mathbb{T}} \beta(2^k |\omega - \omega_m + i\epsilon|) e^{-i\omega T} \rho(\omega) f(\omega, \epsilon) \sqrt{\omega - \omega_m + i\epsilon} d\omega, \quad k \in \mathbb{N}. \quad (6.23)$$

Since the expression under the integral defining F_0 is smooth in ω and in ϵ , there is $C_1 < \infty$ independent on $\epsilon \in (0, 1)$ such that $|F_0(T, \epsilon)| \leq C_1 |T|^{-3/2}$, $T \in \mathbb{Z}$. To estimate $|F_k(T, \epsilon)|$ with $k \in \mathbb{N}$, we first notice that

$$|F_k(T, \epsilon)| \leq C_2 2^{-3k/2}, \quad (6.24)$$

with $C_2 < \infty$ bounded uniformly for $\epsilon \in (0, 1)$: the factor 2^{-k} comes from the size of the support in ω and $2^{-k/2}$ from the magnitude of the square root. We can also integrate in (6.23) by parts two times in ω (with the aid of the operator $L = iT^{-1}\partial_\omega$ which is the identity when applied to the exponential), getting

$$|F_k(T, \epsilon)| \leq C_3 2^{-3k/2} \left(\frac{2^k}{|T|} \right)^2, \quad (6.25)$$

where $2^k/|T|$ is the contribution from each integration by parts, when ∂_ω could fall onto either on β or on the square root (producing a factor of 2^k) or onto $\rho(\omega)f(\omega, \epsilon)$, and $C_3 < \infty$ does not depend on $\epsilon \in (0, 1)$ and $T \in \mathbb{Z}$. Thus, taking into account (6.24) and (6.25), we get the estimate

$$\begin{aligned} |F(T, \epsilon)| &\leq |F_0(T, \epsilon)| + \sum_{k \in \mathbb{N}} |F_k(T, \epsilon)| \\ &\leq C_1 |T|^{-\frac{3}{2}} + \sum_{k \in \mathbb{N}: 2^k > |T|} C_2 2^{-\frac{3k}{2}} + \sum_{k \in \mathbb{N}: 2^k \leq |T|} C_3 2^{\frac{k}{2}} |T|^{-2} \leq C |T|^{-\frac{3}{2}}, \end{aligned}$$

valid for all $T \in \mathbb{Z}$; above, $C < \infty$ does not depend on $\epsilon \in (0, 1)$. It follows that $|\mathcal{F}^{-1}[\frac{\rho(\omega)}{\mathcal{G}(\omega+i\epsilon)}](T)| \leq C|T|^{-3/2}$, for any $\epsilon \in (0, 1)$, $T \in \mathbb{Z}$. By the dominated convergence theorem (albeit with $T \in \mathbb{Z}$), the convergence (6.20) follows. \square

Now we can extend a version of Lemma 6.1 with $X = 0$ to the supports of \tilde{g} and $\tilde{\zeta}_0$, which by Lemmas 6.2 and 6.4 could include endpoints of the continuous spectrum, $\partial\Omega_c$.

Lemma 6.9. *The solution to the stationary problem (6.1) satisfies*

$$r(\omega)\tilde{\zeta}_0(\omega) = \tau^2 \tilde{g}(\omega), \quad \omega \in \mathbb{T}. \quad (6.26)$$

Proof. By (5.22),

$$\frac{1}{\mathcal{G}_0(\omega)} \tilde{\varphi}_0(\omega) = \tau^2 \tilde{f}(\omega), \quad \omega \in \mathbb{C}^+ \bmod 2\pi. \quad (6.27)$$

Now, similarly to Lemma 6.1, the proof follows from taking the limit $\text{Im } \omega \rightarrow 0+$ in (6.27); let us provide details. Due to the convergence (6.5), the right hand-side of (6.27) converges to $\tau^2 \tilde{g}(\omega)$, $\omega \in \mathbb{R}$; let us now consider the left-hand side of (6.27). The convergence (6.7) also takes place in the space of $l^2(\mathbb{Z}^n)$ -valued quasimeasures, $\mathcal{Q}(\mathbb{Z}, l^2(\mathbb{Z}^n))$. Besides, there is a convergence $\frac{1}{\mathcal{G}_0(\omega+i\epsilon)} \rightarrow r(\omega)$ as $\epsilon \rightarrow 0+$ in the space of multipliers in $\mathcal{Q}(\mathbb{T} \setminus \Sigma')$ stated in Lemma 6.7. Noticing that, by Lemma 6.4, $\text{supp } \tilde{\zeta}_0 \subset \overline{\Omega}_0 \cup \overline{\Omega}_\pi \subset \mathbb{T} \setminus \Sigma'$, Lemma 6.6 on multipliers in the space of quasimeasures allows us to conclude that the left-hand side of (6.27) converges to $\frac{1}{\mathcal{G}_0(\omega)} \tilde{\zeta}_0(\omega)$, $\omega \in \mathbb{T}$, and the statement of the lemma follows. \square

We define the “sharp” operation \sharp on $\mathcal{D}'(\mathbb{T})$ by

$$f^\sharp(\omega) := \bar{f}(-\omega), \quad \omega \in \mathbb{T}. \quad (6.28)$$

Remark 6.10. For $F \in l(\mathbb{Z})$, one has $(\tilde{F})^\sharp = \tilde{\bar{F}}$.

Now we will use the spectral representation stated in Lemma 6.9 to obtain the following fundamental relation satisfied by ζ_0 , which we call *the spectral relation*. In the next section we will apply our version of the Titchmarsh convolution theorem to this relation, proving that the ω -support of $\tilde{\zeta}$ consists of one, two, or four frequencies.

Lemma 6.11.

$$(r\tilde{\zeta}_0) * (\tilde{\zeta}_0^\sharp i \sin \omega) + (r\tilde{\zeta}_0^\sharp) * (\tilde{\zeta}_0 i \sin \omega) = -\tau^2 \widetilde{W(|\zeta_0^T|^2)} i \sin \omega, \quad \omega \in \mathbb{T}. \quad (6.29)$$

Proof. Denote by b^T the coefficient appearing in (3.2):

$$b^T = \begin{cases} \frac{W(|\zeta_0^{T+1}|^2) - W(|\zeta_0^{T-1}|^2)}{|\zeta_0^{T+1}|^2 - |\zeta_0^{T-1}|^2} & \text{if } |\zeta_0^{T+1}|^2 \neq |\zeta_0^{T-1}|^2, \\ W'(|\zeta_0^{T+1}|^2) & \text{if } |\zeta_0^{T+1}|^2 = |\zeta_0^{T-1}|^2. \end{cases} \quad (6.30)$$

Then (3.2) takes the form $g^T = -b^T \frac{\zeta_0^{T+1} + \zeta_0^{T-1}}{2}$, or, on the Fourier transform side,

$$\tilde{g}(\omega) = -\tilde{b} * (\tilde{\zeta}_0(\omega) \cos \omega), \quad \omega \in \mathbb{T}. \quad (6.31)$$

Here and below, we are using the identities

$$\frac{\widetilde{f^{T+1}}(\omega) + \widetilde{f^{T-1}}(\omega)}{2} = \tilde{f}(\omega) \cos \omega, \quad \frac{\widetilde{f^{T+1}}(\omega) - \widetilde{f^{T-1}}(\omega)}{2} = -i\tilde{f}(\omega) \sin \omega, \quad (6.32)$$

valid for any $\tilde{f} \in \mathcal{D}'(\mathbb{T})$, which follow from the definition of the Fourier transform (5.5). Substituting (6.31) into (6.26), we obtain the relation

$$r(\omega)\tilde{\zeta}_0(\omega) = -\tau^2\tilde{b} * (\tilde{\zeta}_0 \cos \omega), \quad \omega \in \mathbb{T}. \quad (6.33)$$

We note that $r^\sharp(\omega) := \overline{r(-\omega)} = \overline{r(\omega)} = r(\omega)$ since $r(\omega)$ is even and real-valued on the support of $\tilde{\zeta}_0(\omega)$ by Lemma 6.7, and similarly $\tilde{b}^\sharp(\omega) = \overline{\tilde{b}(-\omega)} = \tilde{b}(\omega) = \tilde{b}(\omega)$ since \tilde{b}^T is real-valued by (6.30). Hence, by Lemma 6.9 and (6.31), we have:

$$r(\omega)\tilde{\zeta}_0^\sharp(\omega) = -\tau^2\tilde{b} * (\tilde{\zeta}_0^\sharp \cos \omega), \quad \omega \in \mathbb{T}. \quad (6.34)$$

We take the convolution of (6.33) with $\tilde{\zeta}_0^\sharp(\omega)i \sin \omega$, the convolution of (6.34) with $\tilde{\zeta}_0(\omega)i \sin \omega$, and add them up:

$$\begin{aligned} & (r\tilde{\zeta}_0) * (\tilde{\zeta}_0^\sharp i \sin \omega) + (r\tilde{\zeta}_0^\sharp) * (\tilde{\zeta}_0 i \sin \omega) \\ &= -\tau^2\tilde{b} * [(\tilde{\zeta}_0 \cos \omega) * (\tilde{\zeta}_0^\sharp i \sin \omega) + (\tilde{\zeta}_0^\sharp \cos \omega) * (\tilde{\zeta}_0 i \sin \omega)]. \end{aligned} \quad (6.35)$$

Using the identity $\sin(\omega - \sigma) \cos \sigma + \cos(\omega - \sigma) \sin \sigma = \sin \omega$, we rewrite the expression in brackets as $(\tilde{\zeta}_0^\sharp * \tilde{\zeta}_0)i \sin \omega$, which, due to the identities (6.32), is the Fourier transform of $-\frac{1}{2}(|\zeta_0^{T+1}|^2 - |\zeta_0^{T-1}|^2)$. This leads to

$$(r\tilde{\zeta}_0) * (\tilde{\zeta}_0^\sharp i \sin \omega) + (r\tilde{\zeta}_0^\sharp) * (\tilde{\zeta}_0 i \sin \omega) = \frac{\tau^2}{2}\tilde{b} * (\widetilde{|\zeta_0^{T+1}|^2} - \widetilde{|\zeta_0^{T-1}|^2}).$$

As follows from (6.30), the right-hand side of the above relation is the Fourier transform of $\frac{\tau^2}{2}(W(|\zeta_0^{T+1}|^2) - W(|\zeta_0^{T-1}|^2))$. Taking into account the identity

$$\frac{1}{2}(W(\widetilde{|\zeta_0^{T+1}|^2}) - W(\widetilde{|\zeta_0^{T-1}|^2})) = -W(\widetilde{|\zeta_0^T|^2})i \sin \omega,$$

we rewrite (6.35) in the desired form

$$(r\tilde{\zeta}_0) * (\tilde{\zeta}_0^\sharp i \sin \omega) + (r\tilde{\zeta}_0^\sharp) * (\tilde{\zeta}_0 i \sin \omega) = -\tau^2 W(\widetilde{|\zeta_0^T|^2})i \sin \omega.$$

□

7 Nonlinear spectral analysis of omega-limit trajectories

We are going to prove our main result, reducing the spectrum of ζ_0^T to at most four points. Denote

$$\mathring{\mathbb{T}} = \mathbb{T} \setminus \left\{ \pm \frac{\pi}{2} \right\}, \quad \mathcal{D}'(\mathring{\mathbb{T}}) = \{f \in \mathcal{D}'(\mathbb{T}) : \text{supp } f \subset \mathring{\mathbb{T}}\}, \quad (7.1)$$

Definition 7.1. For $f \in \mathcal{D}'(\mathring{\mathbb{T}})$, define

$$\text{supp}_{\text{mod } \pi} f = \left\{ p \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) : \text{either } p \in \text{supp } f \text{ or } p + \pi \in \text{supp } f \right\}.$$

Remark 7.2. For $f \in \mathcal{D}'(\mathring{\mathbb{T}})$, the convex hull of $\text{supp}_{\text{mod } \pi} f$, which is denoted by $\text{c.h. supp}_{\text{mod } \pi} f$, is the smallest closed interval $I \subset (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\text{supp } f \subset I \cup (\pi + I)$.

For $y \in \mathbb{T}$, let S_y be the shift operator acting on $\mathcal{D}'(\mathbb{T})$: for $f \in \mathcal{D}'(\mathbb{T})$,

$$S_y f(\omega) = f(\omega + y), \quad \omega \in \mathbb{T}. \quad (7.2)$$

Note that $S_\pi : \mathcal{D}'(\mathring{\mathbb{T}}) \rightarrow \mathcal{D}'(\mathring{\mathbb{T}})$.

Definition 7.3. For $\kappa > 0$, define the following subsets of $\mathcal{D}'(\mathring{\mathbb{T}})$:

$$\begin{aligned} \mathbf{L}_\kappa^\pm &= \left\{ f \in \mathcal{D}'(\mathring{\mathbb{T}}) : f \neq 0, f = \pm S_\pi f \text{ on } \left(-\frac{\pi}{2}, \inf_{\text{mod } \pi} \text{supp } f + \kappa \right) \right\}, \\ \mathbf{R}_\kappa^\pm &= \left\{ f \in \mathcal{D}'(\mathring{\mathbb{T}}) : f \neq 0, f = \pm S_\pi f \text{ on } \left(\sup_{\text{mod } \pi} \text{supp } f - \kappa, \frac{\pi}{2} \right) \right\}. \end{aligned}$$

We use the following version of the Titchmarsh convolution theorem on the circle.

Theorem 7.4 (Titchmarsh theorem for distributions on the circle). *Let $f, g \in \mathcal{D}'(\mathring{\mathbb{T}})$ satisfy $\text{supp}_{\text{mod } \pi} f + \text{supp}_{\text{mod } \pi} g \subset (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $f * g \in \mathcal{D}'(\mathring{\mathbb{T}})$, and for each $\kappa > 0$ the following statements (A) and (B) are equivalent:*

$$(A) \quad \inf_{\text{mod } \pi} \text{supp } f * g \geq \inf_{\text{mod } \pi} \text{supp } f + \inf_{\text{mod } \pi} \text{supp } g + \kappa;$$

$$(B) \quad \text{Either } f \in \mathbf{L}_\kappa^-, g \in \mathbf{L}_\kappa^+, \text{ or } f \in \mathbf{L}_\kappa^+, g \in \mathbf{L}_\kappa^-.$$

Similarly, the following statements (A') and (B') are equivalent:

$$(A') \quad \sup_{\text{mod } \pi} \text{supp } f * g \leq \sup_{\text{mod } \pi} \text{supp } f + \sup_{\text{mod } \pi} \text{supp } g - \kappa;$$

$$(B') \quad \text{Either } f \in \mathbf{R}_\kappa^-, g \in \mathbf{R}_\kappa^+, \text{ or } f \in \mathbf{R}_\kappa^+, g \in \mathbf{R}_\kappa^-.$$

Corollary 7.5. *Let $f \in \mathcal{D}'(\mathring{\mathbb{T}})$ be such that $\text{supp } f \subset (-\frac{\pi}{2N}, \frac{\pi}{2N}) \cup (\pi - \frac{\pi}{2N}, \pi + \frac{\pi}{2N})$, where $N \in \mathbb{N}$. Then*

$$\text{c.h. sup}_{\text{mod } \pi} \underbrace{f * \dots * f}_N = N \text{ c.h. sup}_{\text{mod } \pi} f.$$

These results in a slightly different formulation are proved in Appendix B (see Theorems B.1, B.3).

Lemma 7.6. *Under conditions of Proposition 3.3,*

$$\text{c.h. sup}_{\text{mod } \pi} \left(W(\widetilde{|\zeta_0^T|^2}) \sin \omega \right) \subset \text{c.h. sup}_{\text{mod } \pi} \left(\widetilde{|\zeta_0^T|^2} \right). \quad (7.3)$$

Proof. According to Lemma 6.11, it suffices to prove that

$$\text{c.h. sup}_{\text{mod } \pi} \left((r\tilde{\zeta}_0) * (\tilde{\zeta}_0^\sharp(\omega) i \sin \omega) + (r\tilde{\zeta}_0^\sharp) * (\tilde{\zeta}_0(\omega) i \sin \omega) \right) \subset \text{c.h. sup}_{\text{mod } \pi} \tilde{\zeta}_0^\sharp * \tilde{\zeta}_0. \quad (7.4)$$

Set

$$h_1(\omega) = r(\omega)\tilde{\zeta}_0(\omega) + \tilde{\zeta}_0(\omega)i \sin \omega, \quad h_2(\omega) = r(\omega)\tilde{\zeta}_0^\sharp(\omega), \quad h_3(\omega) = \tilde{\zeta}_0^\sharp(\omega)i \sin \omega. \quad (7.5)$$

Note that $\overline{i \sin(-\omega)} = i \sin \omega$ and $\overline{r(-\omega)} = r(\omega)$ for $\omega \in \overline{\mathcal{I}_{00}} \cup \overline{\mathcal{I}_{0\pi}}$ by (6.19), hence

$$h_1^\sharp(\omega) = r(\omega)\tilde{\zeta}_0^\sharp(\omega) + \tilde{\zeta}_0^\sharp(\omega)i \sin \omega, \quad h_2^\sharp(\omega) = r(\omega)\tilde{\zeta}_0^\sharp(\omega), \quad h_3^\sharp(\omega) = \tilde{\zeta}_0^\sharp(\omega)i \sin \omega.$$

There is the identity

$$(r\tilde{\zeta}_0) * (\tilde{\zeta}_0^\sharp i \sin \omega) + (r\tilde{\zeta}_0^\sharp) * (\tilde{\zeta}_0 i \sin \omega) = h_1^\sharp * h_1 - h_2^\sharp * h_2 - h_3^\sharp * h_3.$$

Therefore, to prove the inclusion (7.4), it suffices to prove that

$$\text{c.h. sup}_{\text{mod } \pi} (h_j^\sharp * h_j) \subset \text{c.h. sup}_{\text{mod } \pi} (\tilde{\zeta}_0^\sharp * \tilde{\zeta}_0), \quad j = 1, 2, 3. \quad (7.6)$$

Set $[a, b] = \text{c.h. sup}_{\text{mod } \pi} \tilde{\zeta}_0 \subset \overline{\mathcal{I}_{00}} \subset (-\frac{\pi}{2}, \frac{\pi}{2})$, with $a \leq b$.

Obviously, $\text{supp } \tilde{\zeta}_0^\# * \tilde{\zeta}_0$ is symmetric (with respect to $\omega = 0$), and $\text{supp } \tilde{\zeta}_0^\# * \tilde{\zeta}_0 \subset [-b + a, b - a]$. We consider the following two cases.

Case 1: c.h. $\text{supp } \tilde{\zeta}_0^\# * \tilde{\zeta}_0 = [-b + a, b - a]$. In this case, the inclusion (7.4) is immediate. Indeed, since c.h. $\text{supp } \tilde{\zeta}_0 = [a, b]$, one has:

$$\text{supp } h_j \subset [a, b] \cup [\pi + a, \pi + b], \quad \text{supp } h_j^\# \subset [-b, -a] \cup [\pi - b, \pi - a]; \quad 1 \leq j \leq 3.$$

Therefore, $\text{supp}(h_j^\# * h_j) \subset [-b + a, b - a] \cup [\pi - b + a, \pi + b - a]$. Since these two subsets of \mathbb{T} do not intersect, one concludes that c.h. $\text{supp } (h_j^\# * h_j) \subset [-b + a, b - a]$.

Case 2: for some $\kappa > 0$

$$\text{c.h. } \text{supp } \tilde{\zeta}_0^\# * \tilde{\zeta}_0 = [-b + a + \kappa, b - a - \kappa].$$

By Theorem 7.4, we have $\tilde{\zeta}_0 \in \mathbf{L}_\kappa^\sigma$ and $\tilde{\zeta}_0^\# \in \mathbf{L}_\kappa^{-\sigma}$, with some $\sigma \in \{\pm\}$, and also $\tilde{\zeta}_0 \in \mathbf{R}_\kappa^\rho$ and $\tilde{\zeta}_0^\# \in \mathbf{R}_\kappa^{-\rho}$, with some $\rho \in \{\pm\}$. However, by (6.28), $f \in \mathbf{R}_\kappa^\pm$ implies that $f^\# \in \mathbf{L}_\kappa^\pm$; therefore, $\sigma = -\rho$, so that

$$\text{either } \tilde{\zeta}_0 \in \mathbf{L}_\kappa^+ \cap \mathbf{R}_\kappa^-, \quad \tilde{\zeta}_0^\# \in \mathbf{L}_\kappa^- \cap \mathbf{R}_\kappa^+, \quad (7.7)$$

$$\text{or } \tilde{\zeta}_0 \in \mathbf{L}_\kappa^- \cap \mathbf{R}_\kappa^+, \quad \tilde{\zeta}_0^\# \in \mathbf{L}_\kappa^+ \cap \mathbf{R}_\kappa^-.$$

In (7.5), the multipliers $r(\omega)$ and $\sin \omega$ are π -antiperiodic by (6.19). Hence, the functions $h_j(\omega)$, $j = 1, 2, 3$, defined in (7.5), satisfy the inclusion

$$\text{either } h_j \in \mathbf{L}_\kappa^- \cap \mathbf{R}_\kappa^+, \quad 1 \leq j \leq 3, \quad \text{or } h_j \in \mathbf{L}_\kappa^+ \cap \mathbf{R}_\kappa^-, \quad 1 \leq j \leq 3, \quad (7.8)$$

with the \pm signs opposite to the signs in (7.7). Also, since the multipliers in (7.5) satisfy $r^\#(\omega) = r(\omega)$, $(i \sin \omega)^\# = i \sin \omega$, the functions $h_j^\#$, $j = 1, 2, 3$, satisfy

$$\text{either } h_j^\# \in \mathbf{L}_\kappa^+ \cap \mathbf{R}_\kappa^-, \quad 1 \leq j \leq 3, \quad \text{or } h_j^\# \in \mathbf{L}_\kappa^- \cap \mathbf{R}_\kappa^+, \quad 1 \leq j \leq 3. \quad (7.9)$$

Due to inclusions (7.8) and (7.9), Theorem 7.4 gives

$$\text{c.h. } \text{supp } (h_j^\# * h_j) \subset [-b + a + \kappa, b - a - \kappa].$$

This yields (7.6), finishing the proof. \square

Lemma 7.7. $\text{supp } |\widehat{\zeta_0^T}|^2 \subset \{0; \pi\}$.

Proof. By Lemma 7.6,

$$\text{c.h. } \text{supp } \left(|\widehat{\zeta_0^T}|^{2(p+1)} \sin \omega \right) \subset \bigcup_{1 \leq q \leq p} \text{c.h. } \text{supp } \left(|\widehat{\zeta_0^T}|^{2q} \right), \quad (7.10)$$

since $C_p > 0$ (see Assumption 2.4). At the same time, due to Lemma 6.4, one has c.h. $\text{supp } \tilde{\zeta}_0 \subset [-\omega_m, \omega_m]$, therefore

c.h. $\text{supp } |\widehat{\zeta_0^T}|^2 \subset [-2\omega_m, 2\omega_m]$. Since $\omega_m < \frac{\pi}{4(p+1)}$ by Assumption 2.9, we see that Corollary 7.5 is applicable to

$$\underbrace{|\widehat{\zeta_0^T}|^2 * \dots * |\widehat{\zeta_0^T}|^2}_q = |\widehat{\zeta_0^T}|^{2q},$$

for each q between 1 and $p + 1$; thus, (7.10) shows that c.h. $\text{supp } |\widehat{\zeta_0^T}|^2 \subset \{0\}$, which is equivalent to $\text{supp } |\widehat{\zeta_0^T}|^2 \subset \{0; \pi\}$. \square

Lemma 7.8. *One of the following possibilities takes place:*

(i) $\tilde{\zeta}_0(\omega) = A\delta_a(\omega)$, with $a \in \overline{\Omega_0} \cup \overline{\Omega_\pi}$ and $A \in \mathbb{C}$;

(ii) $\tilde{\zeta}_0(\omega) = A\delta_a(\omega) + A'\delta_{\pi+a}(\omega)$, with $a \in \overline{\Omega_0} \cup \overline{\Omega_\pi}$ and $A, A' \in \mathbb{C}$;

(iii) $\tilde{\zeta}_0(\omega) = A(\delta_a(\omega) + \delta_{\pi+a}(\omega)) + B(\delta_b(\omega) - \delta_{\pi+b}(\omega))$, with $a, b \in \overline{\Omega_0} \cup \overline{\Omega_\pi}$ and $A, B \in \mathbb{C}$.

Remark 7.9. The above lemma is similar to Theorem B.4 in Appendix B.

Proof. By Lemma 6.4, $\text{supp}_{\text{mod } \pi} \tilde{\zeta}_0 \subset \overline{\Omega_0}$; we denote $[a, b] := \text{c.h. sup}_{\text{mod } \pi} \tilde{\zeta}_0 \subset \overline{\Omega_0}$.

If $a = b$, so that $\text{c.h. sup}_{\text{mod } \pi} \tilde{\zeta}_0 = \{a\} \subset \overline{\Omega_0}$, thus $\text{supp}_{\text{mod } \pi} \tilde{\zeta}_0 \subset \{a; \pi + a\}$; this is equivalent to the first or second possibilities stated in the lemma.

Now let us consider the case when $a < b$. By Lemma 7.7, $\text{c.h. sup}_{\text{mod } \pi} \tilde{\zeta}_0^\# * \tilde{\zeta}_0 \subset \{0\}$. Theorem 7.4 is applicable to $\tilde{\zeta}_0^\# * \tilde{\zeta}_0$ since $\text{supp}_{\text{mod } \pi} \tilde{\zeta}_0 \subset \overline{\Omega_0} = [-\omega_m, \omega_m]$, with $\omega_m < \frac{\pi}{4(p+1)}$, $p \geq 1$ (Cf. Assumption 2.9). The inclusion $\text{supp}_{\text{mod } \pi} \tilde{\zeta}_0^\# * \tilde{\zeta}_0 \subset \{0\}$ implies that one can take $\kappa = b - a$ in the statement (A') in Theorem 7.4, therefore either

$$\tilde{\zeta}_0^\# \in \mathbf{R}_{b-a}^+ \quad \text{and} \quad \tilde{\zeta}_0 \in \mathbf{R}_{b-a}^-, \quad \text{hence} \quad \tilde{\zeta}_0 \in \mathbf{L}_{b-a}^+ \cap \mathbf{R}_{b-a}^-, \quad (7.11)$$

or

$$\tilde{\zeta}_0^\# \in \mathbf{R}_{b-a}^- \quad \text{and} \quad \tilde{\zeta}_0 \in \mathbf{R}_{b-a}^+, \quad \text{hence} \quad \tilde{\zeta}_0 \in \mathbf{L}_{b-a}^- \cap \mathbf{R}_{b-a}^+. \quad (7.12)$$

The cases (7.11) and (7.12) are considered in a similar manner. If (7.11) is satisfied, then

$$\tilde{\zeta}_0 = -S_\pi \tilde{\zeta}_0 \quad \text{on} \quad (a, \pi/2) \quad \text{and} \quad \tilde{\zeta}_0 = S_\pi \tilde{\zeta}_0 \quad \text{on} \quad (-\pi/2, b), \quad (7.13)$$

hence $\tilde{\zeta}_0|_{(a,b)} = 0$, implying that

$$\text{supp}_{\text{mod } \pi} \tilde{\zeta}_0 \subset \{a; b; \pi + a; \pi + b\}, \quad (7.14)$$

and moreover

$$\tilde{\zeta}_0 = A(\delta_a + \delta_{\pi+a}) + B(\delta_b - \delta_{\pi+b}), \quad A, B \in \mathbb{C} \setminus 0. \quad (7.15)$$

Note that, due to the boundedness of ζ_0^T (which follows from applying Theorem 2.7 to ψ and then to its omega-limit trajectory ζ), its Fourier transform can not contain the derivatives of δ -functions. We are thus in the framework of the third possibility stated in the lemma.

If, instead, (7.12) is satisfied, we are led to the conclusion

$$\text{supp}_{\text{mod } \pi} \tilde{\zeta}_0 \subset \{a; b; \pi + a; \pi + b\}, \quad (7.16)$$

and moreover

$$\tilde{\zeta}_0 = A(\delta_a - \delta_{\pi+a}) + B(\delta_b + \delta_{\pi+b}), \quad A, B \in \mathbb{C} \setminus 0. \quad (7.17)$$

This again puts us in the framework of the third possibility stated in the lemma. \square

Lemma 7.10. ζ^T is a multifrequency solitary wave with one, two, or four frequencies.

If $n \leq 4$, then one has $\text{supp } \tilde{\zeta} \subset \mathbb{Z}^n \times (\Omega_0 \cup \Omega_\pi)$.

Proof. By Lemma 6.9,

$$\tilde{g}(\omega) = \frac{1}{\tau^2} r(\omega) \tilde{\zeta}_0(\omega), \quad \omega \in \mathbb{T}. \quad (7.18)$$

Therefore, due to Lemma 7.8, $\text{supp } \tilde{g}$ consists of one, two, or four points inside $\overline{\Omega_0} \cup \overline{\Omega_\pi}$. By Lemma 6.1,

$$\tilde{\zeta}_X(\omega) = \frac{\mathcal{G}_X(\omega)}{\mathcal{G}_0(\omega)} \tilde{\zeta}_0(\omega), \quad X \in \mathbb{Z}^n, \quad \omega \in \Omega_0 \cup \Omega_\pi, \quad (7.19)$$

where we took into account that, by Lemma 5.2, $\mathcal{G}_0(\omega) \neq 0$ for $\omega \in \Omega_0 \cup \Omega_\pi$. This implies that

$$\tilde{\zeta}(\omega) = \sum_{j=1}^4 A_j \delta(\omega - \omega_j), \quad \omega_j \in \overline{\Omega_0} \cup \overline{\Omega_\pi}, \quad A_j \in l^2(\mathbb{Z}^n), \quad 1 \leq j \leq 4. \quad (7.20)$$

This implies that ζ^T is a multifrequency solitary wave of the form (2.11).

By Lemma 6.1, for each j such that $\omega_j \notin \partial\Omega_c$, in (7.20) one has $A_j = \tau^2 \mathcal{G}(\omega_j)$. Moreover, substituting (7.20) into (6.1), we see that even if $\omega_j \in \partial\Omega_c$, then one needs to have $A_j = c_j \mathcal{G}(\omega_j)$, with some $c_j \in \mathbb{C}$. At the same time, by Lemma 4.1, if $n \leq 4$, then for $\omega \in \mathbb{T}$ one has $\mathcal{G}(\omega) \in l^2(\mathbb{Z}^n)$ if and only if $\omega \in \Omega_0 \cup \Omega_\pi$. Thus, in the case $n \leq 4$, the requirement that $\zeta^T \in l^2(\mathbb{Z}^n)$ leads to the inclusion $\omega_j \in \Omega_0 \cup \Omega_\pi$, $1 \leq j \leq 4$.

Note that in the case $n \geq 5$, when one could have $\omega_j \in \partial\Omega_c$ for some $1 \leq j \leq 4$, the derivatives of $\delta(\omega - \omega_j)$ do not appear in (7.20) due to the uniform $l^2(\mathbb{Z}^n)$ -bounds on ζ^T . \square

By Lemma 7.10, we know that the set of all omega-limit trajectories consists of multifrequency waves. In Section 8, we will check that the two- and four-frequency solitary waves have the form specified in Proposition 3.3; see Lemma 8.2 and Lemma 8.3 below. This will finish the proof of Proposition 3.3.

Let us complete this section with a simple derivation of the form of four-frequency solitary waves in dimensions $n \leq 4$.

Lemma 7.11. *Let $n \leq 4$. Each four-frequency solitary wave can be represented in the form*

$$\psi_X^T = (1 + (-1)^{T+\Lambda \cdot X})\phi_X e^{-i\omega T} + (1 - (-1)^{T+\Lambda \cdot X})\theta_X e^{-i\omega' T}, \quad X \in \mathbb{Z}^n, \quad T \in \mathbb{Z}, \quad (7.21)$$

with $\phi, \theta \in l^2(\mathbb{Z}^n)$.

Proof. Recall that, by Lemma 3.4, each multifrequency solitary wave is its own omega-limit trajectory; therefore, it is enough to prove that each four-frequency omega-limit trajectory has the form (7.21).

Since $\text{supp } \tilde{\zeta}_X \subset \Omega_0 \cup \Omega_\pi$ by Lemma 7.10, the relation (7.19) could be extended to $\omega \in \mathbb{T}$. To have a four-frequency omega-limit trajectory, $\tilde{\zeta}_0$ is to be given by (7.15). Then (7.19) yields

$$\begin{aligned} \tilde{\zeta}_X &= \frac{\mathcal{G}_X(\omega)}{\mathcal{G}_0(\omega)} (A(\delta_a + \delta_{\pi+a}) + B(\delta_b - \delta_{\pi+b})) \\ &= A \frac{\mathcal{G}_X(a)}{\mathcal{G}_0(a)} (\delta_a + (-1)^{\Lambda \cdot X} \delta_{\pi+a}) + B \frac{\mathcal{G}_X(b)}{\mathcal{G}_0(b)} (\delta_b - (-1)^{\Lambda \cdot X} \delta_{\pi+b}), \end{aligned} \quad (7.22)$$

where we took into account that, by Lemma 5.2 (Cf. (5.15)),

$$\frac{\mathcal{G}_X(\omega + \pi)}{\mathcal{G}_0(\omega + \pi)} = (-1)^{\Lambda \cdot X} \frac{\mathcal{G}_X(\omega)}{\mathcal{G}_0(\omega)}, \quad \omega \in \Omega_0 \cup \Omega_\pi.$$

It follows that

$$\zeta_X^T = A \frac{\mathcal{G}_X(a)}{\mathcal{G}_0(a)} (1 + (-1)^{T+\Lambda \cdot X}) e^{-iaT} + B \frac{\mathcal{G}_X(b)}{\mathcal{G}_0(b)} (1 - (-1)^{T+\Lambda \cdot X}) e^{-ibT},$$

finishing the proof. \square

8 Analysis of solitary wave solutions

Here we discuss in more detail one-, two-, and four-frequency solitary waves, prove that they have the form specified in Proposition 3.3, and construct particular examples.

8.1 One-frequency solitary waves

Lemma 8.1. (i) *If $n \leq 4$, there could only be nonzero solitary waves $\phi e^{-i\omega T}$ with $\phi \in l^2(\mathbb{Z}^n)$ for $\omega \in \Omega_0 \cup \Omega_\pi$, where Ω_0 and Ω_π are defined in (2.14). If $n \geq 5$, there could only be solitary waves for $\omega \in \overline{\Omega_0} \cup \overline{\Omega_\pi}$.*

(ii) *For a particular value $\omega \in \Omega_0 \cup \Omega_\pi$ ($\omega \in \overline{\Omega_0} \cup \overline{\Omega_\pi}$ if $n \geq 5$), there is a nonzero solitary wave if and only if*

$$\frac{1}{\mathcal{G}_0(\omega) \cos \omega} \in \text{Range}(-\tau^2 W'(\lambda)|_{\lambda > 0}). \quad (8.1)$$

(iii) *The component of the solitary manifold which corresponds to one-frequency solitary waves is generically two-dimensional.*

(iv) In the case $n = 1$, the necessary and sufficient criterion for the existence of nonzero solitary waves is

$$\left(0, 2\sqrt{\left(1 + \frac{\tau^2 m^2}{2}\right)^2 - 1}\right] \cap \text{Range}(-\tau^2 W'(\lambda)|_{\lambda > 0}) \neq \emptyset. \quad (8.2)$$

Proof. Let us substitute the Ansatz $\psi_X^T = \phi_X e^{-i\omega T}$, $\omega \in \mathbb{T}$, into (2.3). Using the relations

$$\begin{aligned} \partial_t^2 \phi_X e^{-i\omega T} &= (e^{-i\omega(t+1)} + e^{-i\omega(t-1)} - 2e^{-i\omega})\phi_X = 2(\cos \omega - 1)\phi_X e^{-i\omega T}, \\ f^T &= -W'(|\phi_0|^2)\phi_0 e^{-i\omega T} \cos \omega \end{aligned}$$

(Cf. (2.4)), we see that ϕ_X satisfies

$$2(\cos \omega - 1)\phi_X = \frac{1}{n} \mathbf{D}_X^2 \phi_X - \tau^2 m^2 \phi_X \cos \omega - \delta_{X,0} \tau^2 W'(|\phi_0|^2)\phi_0 \cos \omega.$$

Equivalently, the Fourier transform $\hat{\phi}(\xi) = \sum_{X \in \mathbb{Z}^n} \phi_X e^{-i\xi \cdot X}$ is to satisfy

$$a(\xi, \omega) \hat{\phi}(\xi) = -\tau^2 W'(|\phi_0|^2)\phi_0 \cos \omega, \quad \xi \in \mathbb{T}^n, \quad \omega \in \mathbb{T}. \quad (8.3)$$

Thus,

$$\hat{\phi}(\xi) = \frac{C}{a(\xi, \omega)}, \quad \xi \in \mathbb{T}^n, \quad \text{or, equivalently, } \phi_X = C \mathcal{G}_X(\omega), \quad X \in \mathbb{Z}^n, \quad (8.4)$$

with $C \in \mathbb{C}$. By Lemma 4.1, ϕ is of finite l^2 -norm if and only if $\omega \in \Omega_0 \cup \Omega_\pi$ for $n \leq 4$, and $\omega \in \overline{\Omega_0} \cup \overline{\Omega_\pi}$ for $n \geq 5$. This proves the first statement of the lemma.

Substituting (8.4) into (8.3), we see that $C \neq 0$ is to satisfy the equation

$$1 = -\tau^2 W'(|C \mathcal{G}_0(\omega)|^2) \mathcal{G}_0(\omega) \cos \omega. \quad (8.5)$$

Equation (8.5) admits a solution if and only if the condition (8.1) holds, proving the second statement of the lemma.

For each $\omega \in \Omega_0 \cup \Omega_\pi$, the set of solutions C to equation (8.5) (if it is nonempty) admits the representation $C = a e^{is}$ with $a > 0$ and $s \in \mathbb{T}$; for each particular ω , the set of values a is discrete (under the assumption that $W(\lambda)$ is a polynomial of degree larger than 1). The solitary manifold can be locally parametrized by two parameters, $a > 0$ and $s \in \mathbb{T}$, proving the third statement of the lemma.

Finally, in the case $n = 1$, the computation yields

$$\mathcal{G}_0(\omega) = \frac{1}{2} \int_{\mathbb{T}^1} \frac{1}{\left(1 + \frac{\tau^2 m^2}{2}\right) \cos \omega - \cos \xi_1} \frac{d\xi_1}{2\pi} = \frac{1}{2} \frac{\text{sign} \cos \omega}{\sqrt{\left(1 + \frac{\tau^2 m^2}{2}\right)^2 \cos^2 \omega - 1}}. \quad (8.6)$$

It follows that

$$\frac{1}{\mathcal{G}_0(\omega) \cos \omega} = 2\sqrt{\left(1 + \frac{\tau^2 m^2}{2}\right)^2 - \frac{1}{\cos^2 \omega}}, \quad \omega \in \Omega_0 \cup \Omega_\pi,$$

hence

$$\text{Range}\left(\frac{1}{\mathcal{G}_0(\omega) \cos \omega} \Big|_{\omega \in \Omega_0 \cup \Omega_\pi}\right) = \left(0, 2\sqrt{\left(1 + \frac{\tau^2 m^2}{2}\right)^2 - 1}\right],$$

showing that (8.1) is equivalent to (8.2). \square

8.2 Two-frequency solitary waves

Let us study two-frequency solitary wave solutions. By Lemma 7.8, the two frequencies of a two-frequency solitary wave differ by π , hence we need to consider solitary wave solutions of the form

$$\psi^T = p e^{-i\omega_1 T} + q e^{-i(\omega_1 + \pi)T}, \quad p, q \in l^2(\mathbb{Z}^n),$$

with $p, q \in l^2(\mathbb{Z}^n)$ not identically zero. We have:

$$\psi_0^T = p_0 e^{-i\omega_1 T} + q_0 e^{-i(\omega_1 + \pi)T}, \quad |\psi_0^T|^2 = \alpha + \beta e^{-i\pi T},$$

where

$$\alpha = |p_0|^2 + |q_0|^2, \quad \beta = 2\operatorname{Re}(\bar{p}_0 q_0). \quad (8.7)$$

We can write

$$W'(|\psi_0^T|) = M + e^{-i\pi T} N,$$

with

$$M = \frac{1}{2}(W'(\alpha + \beta) + W'(\alpha - \beta)), \quad N = \frac{1}{2}(W'(\alpha + \beta) - W'(\alpha - \beta)). \quad (8.8)$$

Taking into account that

$$\frac{1}{2}(\psi_0^{T+1} + \psi_0^{T-1}) = p_0 \cos \omega_1 e^{-i\omega_1 T} - q_0 \cos \omega_1 e^{-i(\omega_1 + \pi)T},$$

we see that the Fourier transform of ψ_X^T (with respect to both time and space variables) satisfies

$$a(\xi, \omega) [\hat{p}\delta_{\omega_1} + \hat{q}\delta_{\omega_1 + \pi}] = -2\pi\tau^2 [M\delta_0 + N\delta_\pi] * [(p_0\delta_{\omega_1} - q_0\delta_{\omega_1 + \pi}) \cos \omega_1].$$

Collecting the coefficients at δ_{ω_1} and $\delta_{\omega_1 + \pi}$, we get the equations

$$\begin{cases} a(\xi, \omega_1) \hat{p}(\xi) = -2\pi\tau^2 (Mp_0 - Nq_0) \cos \omega_1, \\ a(\xi, \omega_1 + \pi) \hat{q}(\xi) = -2\pi\tau^2 (Np_0 - Mq_0) \cos \omega_1. \end{cases}$$

Dividing by $a(\xi, \omega)$ (at particular values of ω) and taking the inverse Fourier transform with respect to ξ , we have:

$$\begin{cases} p_X = -2\pi\tau^2 (Mp_0 - Nq_0) \mathcal{G}_X(\omega_1) \cos \omega_1, \\ q_X = -2\pi\tau^2 (Np_0 - Mq_0) \mathcal{G}_X(\omega_1 + \pi) \cos \omega_1. \end{cases} \quad (8.9)$$

By Lemma 5.2, $\mathcal{G}_0(\omega) \neq 0$ for $\omega \in \Omega_0 \cup \Omega_\pi$ ($\omega \in \overline{\Omega_0} \cup \overline{\Omega_\pi}$ if $n \geq 5$); therefore, if either p_0 or q_0 were zero, (8.9) would yield that either $Mp_0 - Nq_0$ or $Np_0 - Mq_0$ is zero, hence either p_X or q_X would be identically zero. Thus, for two-frequency solitary waves, we can assume that both p_0 and q_0 are nonzero. Then equations (8.9) lead to

$$1 = -2\pi\tau^2 (M - N \frac{q_0}{p_0}) \mathcal{G}_0(\omega_1) \cos \omega_1, \quad 1 = 2\pi\tau^2 (N \frac{p_0}{q_0} - M) \mathcal{G}_0(\omega_1) \cos \omega_1. \quad (8.10)$$

We took into account that $\mathcal{G}_0(\omega_1 + \pi) = -\mathcal{G}_0(\omega_1)$ (Cf. Lemma 5.2). Relations (8.10) are consistent if $\sigma := \frac{q_0}{p_0} = \pm 1$ and

$$1 + 2\pi\tau^2 (M - \sigma N) \mathcal{G}_0(\omega_1) \cos \omega_1 = 0. \quad (8.11)$$

Now we can prove the following lemma.

Lemma 8.2. (i) *The component of the solitary manifold which corresponds to two-frequency solitary waves is generically two-dimensional.*

(ii) *Each two-frequency solitary wave can be represented in the form*

$$\psi_X^T = (1 + (-1)^{T+\Lambda \cdot X} \sigma) \phi_X e^{-i\omega T}, \quad X \in \mathbb{Z}^n, \quad T \in \mathbb{Z}, \quad (8.12)$$

with $\sigma \in \{\pm 1\}$, $\Lambda = (1, \dots, 1) \in \mathbb{Z}^n$, and $\phi \in l^2(\mathbb{Z}^n)$.

Proof. Let us choose $p_0 \in \mathbb{C} \setminus 0$ and $\sigma = \pm 1$, and set $q_0 = \sigma p_0$. Then, by (8.7), $\beta = \sigma\alpha$, and the relations (8.8) give $M - \sigma N = W'(\alpha - \sigma\beta) = W'(0)$. The relation (8.11) takes the form

$$1 + 2\pi\tau^2 W'(0) \mathcal{G}_0(\omega_1) \cos \omega_1 = 0,$$

allowing us to determine ω_1 (if $W'(0) \geq 0$, no such ω_1 exists). Thus, the corresponding component of the solitary manifold is generically of dimension 2.

To prove the second statement of the lemma, we notice that, by (8.9),

$$\begin{aligned} p_X &= -2\pi\tau^2 p_0 (M - \sigma N) \mathcal{G}_X(\omega_1) \cos \omega_1, \\ q_X &= -2\pi\tau^2 \sigma p_0 (\sigma N - M) \mathcal{G}_X(\omega_1 + \pi) \cos \omega_1 = (-1)^{\Lambda \cdot X} \sigma p_X. \end{aligned}$$

In the last equality, we took into account Lemma 5.2 (Cf. (5.15)). This finishes the proof. \square

8.3 Four-frequency solitary waves

By Lemma 7.8, it is enough to consider four-frequency solitary waves of the form

$$\psi^T = pe^{-i\omega_1 T} + qe^{-i(\omega_1 + \pi)T} + re^{-i\omega_2 T} + se^{-i(\omega_2 + \pi)T}, \quad (8.13)$$

with $\omega_1 \not\equiv \omega_2 \pmod{\pi}$ and with $p, q, r, s \in l^2(\mathbb{Z}^n)$. By Lemma 7.8, we can also assume that

$$q_0 = p_0, \quad s_0 = -r_0; \quad (8.14)$$

then

$$\psi_0^T = p_0(e^{-i\omega_1 T} + e^{-i(\omega_1 + \pi)T}) + r_0(e^{-i\omega_2 T} - e^{-i(\omega_2 + \pi)T}), \quad (8.15)$$

$$|\psi_0^T|^2 = 2(|p_0|^2 + |r_0|^2) + 2(|p_0|^2 - |r_0|^2)e^{-i\pi T} = \alpha + \beta e^{-i\pi T}, \quad (8.16)$$

where

$$\alpha = 2(|p_0|^2 + |r_0|^2), \quad \beta = 2(|p_0|^2 - |r_0|^2). \quad (8.17)$$

Using (8.15), we derive

$$\begin{aligned} & \frac{1}{2}(\psi_0^{T+1} + \psi_0^{T-1}) \\ &= p_0 \cos \omega_1 e^{-i\omega_1 T} - p_0 \cos \omega_1 e^{-i(\omega_1 + \pi)T} + r_0 \cos \omega_2 e^{-i\omega_2 T} + r_0 \cos \omega_2 e^{-i(\omega_2 + \pi)T}, \end{aligned}$$

and for its Fourier transform we have

$$\begin{aligned} & \mathcal{F}\left[\frac{1}{2}(\psi_0^{T+1} + \psi_0^{T-1})\right](\omega) \\ &= 2\pi p_0(\delta_{\omega_1}(\omega) - \delta_{\omega_1 + \pi}(\omega)) \cos \omega_1 + 2\pi r_0(\delta_{\omega_2}(\omega) + \delta_{\omega_2 + \pi}(\omega)) \cos \omega_2. \end{aligned}$$

We have $W'(|\psi_0^T|^2) = M + e^{-i\pi T}N$, $T \in \mathbb{Z}$, where M and N are given by

$$M = \frac{1}{2}(W'(\alpha + \beta) + W'(\alpha - \beta)), \quad N = \frac{1}{2}(W'(\alpha + \beta) - W'(\alpha - \beta)), \quad (8.18)$$

hence $\mathcal{F}[W'(|\psi_0^T|^2)](\omega) = 2\pi(M\delta(\omega) + N\delta_\pi(\omega))$. Thus, the Fourier transform of ψ_X^T (with respect to both time and space variables) satisfies the following relation:

$$\begin{aligned} & a(\xi, \omega)[\hat{p}\delta_{\omega_1} + \hat{q}\delta_{\omega_1 + \pi} + \hat{r}\delta_{\omega_2} + \hat{s}\delta_{\omega_2 + \pi}] \\ &= -2\pi\tau^2[M\delta_0 + N\delta_\pi] * [p_0(\delta_{\omega_1} - \delta_{\omega_1 + \pi}) \cos \omega_1 + r_0(\delta_{\omega_2} + \delta_{\omega_2 + \pi}) \cos \omega_2]. \end{aligned}$$

In this equation, δ -functions are functions of $\omega \in \mathbb{T}$; the functions $\hat{p}, \hat{q}, \hat{r}, \hat{s}$ (Fourier transforms of $p, q, r, s \in l^2(\mathbb{Z}^n)$) depend on $\xi \in \mathbb{T}^n$, and the convolution in the right-hand side is with respect to ω . Collecting the coefficients at δ_{ω_1} , $\delta_{\omega_1 + \pi}$, δ_{ω_2} , and $\delta_{\omega_2 + \pi}$, we rewrite the above equation as the following system:

$$\begin{cases} a(\xi, \omega_1)\hat{p}(\xi) = -2\pi\tau^2 p_0(M - N) \cos \omega_1, \\ a(\xi, \omega_1 + \pi)\hat{q}(\xi) = -2\pi\tau^2 p_0(N - M) \cos \omega_1, \\ a(\xi, \omega_2)\hat{r}(\xi) = -2\pi\tau^2 r_0(M + N) \cos \omega_2, \\ a(\xi, \omega_2 + \pi)\hat{s}(\xi) = -2\pi\tau^2 r_0(N + M) \cos \omega_2. \end{cases} \quad (8.19)$$

Dividing each of these equations by $a(\xi, \omega)$ (taken at the appropriate value of ω), taking the inverse Fourier transform with respect to ξ and using the relation $\hat{\mathcal{G}}(\xi, \omega) = \frac{1}{a(\xi, \omega)}$ (Cf. (5.12)), we have:

$$\begin{cases} p_X = -2\pi\tau^2 p_0(M - N)\mathcal{G}_X(\omega_1) \cos \omega_1, \\ q_X = -2\pi\tau^2 p_0(N - M)\mathcal{G}_X(\omega_1 + \pi) \cos \omega_1, \\ r_X = -2\pi\tau^2 r_0(M + N)\mathcal{G}_X(\omega_2) \cos \omega_2, \\ s_X = -2\pi\tau^2 r_0(N + M)\mathcal{G}_X(\omega_2 + \pi) \cos \omega_2. \end{cases} \quad (8.20)$$

Taking into account that, by Lemma 5.2 (Cf. (5.15)), one has $\mathcal{G}_X(\omega + \pi) = -(-1)^{\Lambda \cdot X} \mathcal{G}_X(\omega)$ for $\omega \in \Omega_0 \cup \Omega_\pi$, we rewrite (8.20) as

$$\begin{cases} p_X = -2\pi\tau^2 p_0(M - N)\mathcal{G}_X(\omega_1) \cos \omega_1, \\ q_X = -2\pi\tau^2 p_0(M - N)(-1)^{\Lambda \cdot X} \mathcal{G}_X(\omega_1) \cos \omega_1, \\ r_X = -2\pi\tau^2 r_0(M + N)\mathcal{G}_X(\omega_2) \cos \omega_2, \\ s_X = 2\pi\tau^2 r_0(M + N)(-1)^{\Lambda \cdot X} \mathcal{G}_X(\omega_2) \cos \omega_2. \end{cases} \quad (8.21)$$

To have $p_0 \neq 0$, the first equation leads to the requirement

$$1 + 2\pi\tau^2(M - N)\mathcal{G}_0(\omega_1) \cos \omega_1 = 0. \quad (8.22)$$

Similarly, to have $r_0 \neq 0$, the third equation requires that

$$1 + 2\pi\tau^2(M + N)\mathcal{G}_0(\omega_2) \cos \omega_2 = 0. \quad (8.23)$$

Note that the second and the fourth equations from (8.21) together with (8.22) and (8.23) lead to $q_0 = p_0$ and $s_0 = -r_0$, in consistency with (8.14).

To construct an example of four-frequency solitary waves, one fixes $\omega_1, \omega_2 \in \Omega_0$ and determines M, N from (8.22) and (8.23). Then one takes arbitrary nonzero p_0 and r_0 , defines α and β from (8.17), and chooses a polynomial $W(s)$ such that (8.18) is satisfied.

Lemma 8.3. (i) *The component of the solitary manifold which corresponds to four-frequency solitary waves is generically four-dimensional.*

(ii) *Each four-frequency solitary wave can be represented in the form*

$$\psi_X^T = (1 + (-1)^{T+\Lambda \cdot X})\phi_X e^{-i\omega T} + (1 - (-1)^{T+\Lambda \cdot X})\theta_X e^{-i\omega' T}, \quad X \in \mathbb{Z}^n, \quad T \in \mathbb{Z}, \quad (8.24)$$

with $\phi, \theta \in l^2(\mathbb{Z}^n)$.

Proof. As follows from the above discussion, once we have one solitary wave of this type, we can vary ω_1 and ω_2 . Then the relations (8.22), (8.23) determine M and N . The relations (8.18) determine α and β , and the relations (8.17) determine $|p_0|$ and $|r_0|$; then p_0 and r_0 are known up to (mutually independent) unitary factors. Thus, locally the component of the solitary manifold corresponding to four-frequency solitary waves is parametrized by four variables.

The second statement of the lemma follows from (8.13) after noticing that, by (8.21), one has

$$q_X = (-1)^{\Lambda \cdot X} p_X, \quad s_X = -(-1)^{\Lambda \cdot X} r_X.$$

□

We finished studying the structure of multifrequency solitary wave solutions. Now the proof of Proposition 3.3 is complete.

A Well-posedness for nonlinear wave equation in discrete space-time

A.1 Continuous case

Let us first consider the $U(1)$ -invariant nonlinear wave equation

$$\ddot{\psi}(x, t) = \Delta \psi(x, t) - 2\partial_\lambda v(x, |\psi(x, t)|^2) \psi(x, t), \quad x \in \mathbb{R}^n, \quad (A.1)$$

where $\psi(x, t) \in \mathbb{C}$ and $v(x, \lambda)$ is such that $v \in C(\mathbb{R}^n \times \mathbb{R})$ and $v(x, \cdot) \in C^2(\mathbb{R})$ for each $x \in \mathbb{R}^n$. Equation (A.1) can be written in the Hamiltonian form, with the Hamiltonian

$$\mathcal{E}(\psi, \dot{\psi}) = \int_{\mathbb{R}^n} \left[\frac{|\dot{\psi}|^2}{2} + \frac{|\nabla \psi|^2}{2} + v(x, |\psi(x, t)|^2) \right] dx. \quad (A.2)$$

The value of the Hamiltonian functional \mathcal{E} and the value of the charge functional

$$\mathcal{Q}(\psi, \dot{\psi}) = \frac{i}{2} \int_{\mathbb{R}^n} (\bar{\psi} \dot{\psi} - \dot{\bar{\psi}} \psi) dx \quad (A.3)$$

are formally conserved for solutions to (A.1). A particular case of (A.1) is the nonlinear Klein-Gordon equation, with $v(x, \lambda) = \frac{m^2}{2}\lambda + z(x, \lambda)$, with $m > 0$:

$$\ddot{\psi} = \Delta\psi - m^2\psi - 2\partial_\lambda z(x, |\psi|^2)\psi, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}. \quad (\text{A.4})$$

If $z(x, \lambda) \geq 0$ for all $x \in \mathbb{R}^n$, $\lambda \geq 0$, then the conservation of the energy

$$\int_{\mathbb{R}^n} \left[\frac{|\dot{\psi}|^2}{2} + \frac{|\nabla\psi|^2}{2} + \frac{m^2|\psi|^2}{2} + z(x, |\psi|^2) \right] dx$$

yields an a priori estimate on the norm of the solution:

$$\int_{\mathbb{R}^n} |\psi(x, t)|^2 dx \leq \frac{2}{m^2} \mathcal{E}(\psi|_{t=0}, \dot{\psi}|_{t=0}). \quad (\text{A.5})$$

A.2 Finite difference approximation

Let us now describe the discretized equation. Let $(X, T) \in \mathbb{Z}^n \times \mathbb{Z}$ denote a point of the space-time lattice. We will always indicate the temporal dependence by superscripts and the spatial dependence by subscripts. Fix $\varepsilon > 0$, and let $V_X(\lambda) = v(\varepsilon X, \lambda)$ be a function on $\mathbb{Z}^n \times \mathbb{R}$, so that $V_X \in C^2(\mathbb{R})$ for each $X \in \mathbb{Z}^n$. For $\lambda, \mu \in \mathbb{R}$ and $X \in \mathbb{Z}^n$, we introduce

$$B_X(\lambda, \mu) := \begin{cases} \frac{V_X(\lambda) - V_X(\mu)}{\lambda - \mu}, & \lambda \neq \mu, \\ \partial_\lambda V_X(\lambda), & \lambda = \mu. \end{cases} \quad (\text{A.6})$$

We consider the Vazquez-Strauss finite-difference scheme for (A.1) [SV78]:

$$\begin{aligned} \frac{\psi_X^{T+1} - 2\psi_X^T + \psi_X^{T-1}}{\tau^2} &= \sum_{j=1}^n \frac{\psi_{X+e_j}^T - 2\psi_X^T + \psi_{X-e_j}^T}{\varepsilon^2} \\ &\quad - B_X(|\psi_X^{T+1}|^2, |\psi_X^{T-1}|^2)(\psi_X^{T+1} + \psi_X^{T-1}), \end{aligned} \quad (\text{A.7})$$

where $\psi_X^T \in \mathbb{C}$ is defined on the lattice $(X, T) \in \mathbb{Z}^n \times \mathbb{Z}$. Above,

$$e_1 = (1, 0, 0, 0, \dots) \in \mathbb{Z}^n, \quad e_2 = (0, 1, 0, 0, \dots) \in \mathbb{Z}^n, \quad \text{etc.} \quad (\text{A.8})$$

The continuous limit of (A.7) is given by (A.1), with εX corresponding to $x \in \mathbb{R}^n$ and τT corresponding to $t \in \mathbb{R}$. Since $\partial_\lambda V_X(\lambda) = B_X(\lambda, \lambda)$, the continuous limit of the last term in the right-hand side of (A.7) coincides with the right-hand side in (A.1).

An advantage of the Strauss-Vazquez finite-difference scheme (A.7) over other energy-preserving schemes discussed in [LVQ95, Fur01] is that it is *explicit*: at the moment $T + 1$ the relation (A.7) only involves the function ψ at the point X , allowing for a simple realization of the solution algorithm even in higher dimensional case.

A.3 Well-posedness

We will denote by ψ^T the function ψ defined on the lattice $(X, T) \in \mathbb{Z}^n \times \mathbb{Z}$ at the moment $T \in \mathbb{Z}$.

Theorem A.1 (Existence of solutions). *Assume that*

$$k_1 := \inf_{X \in \mathbb{Z}^n, \lambda \geq 0} \partial_\lambda V_X(\lambda) > -\infty. \quad (\text{A.9})$$

Define

$$\tau_1 = \begin{cases} \sqrt{-1/k_1}, & k_1 < 0; \\ +\infty, & k_1 \geq 0. \end{cases}$$

Then for any $\tau \in (0, \tau_1)$ and any $\varepsilon > 0$ there exists a global solution ψ^T , $T \in \mathbb{Z}$, to the Cauchy problem for equation (A.7) with arbitrary initial data ψ^0, ψ^1 (which stand for ψ^T at $T = 0$ and $T = 1$).

Moreover, if $(\psi^0, \psi^1) \in l^2(\mathbb{Z}^n) \times l^2(\mathbb{Z}^n)$, one has $\psi^T \in l^2(\mathbb{Z}^n)$ for all $T \in \mathbb{Z}$.

Note that we do not claim in this theorem that $\|\psi^T\|_{l^2(\mathbb{Z}^n)}$ is uniformly bounded for all $T \in \mathbb{Z}$. For the a priori estimates on $\|\psi^T\|_{l^2(\mathbb{Z}^n)}$, see Theorem A.8 below.

One can readily check that any X -independent polynomial potential of the form

$$V_X(\lambda) = V(\lambda) = \sum_{q=0}^p C_q \lambda^{q+1}, \quad p \in \mathbb{N}, \quad C_q \in \mathbb{R} \quad \text{for } 0 \leq q \leq p, \quad C_p > 0 \quad (\text{A.10})$$

satisfies (A.9). Note that since $\lim_{\lambda \rightarrow +\infty} V(\lambda) = +\infty$, this potential is confining.

Theorem A.2 (Uniqueness and continuous dependence on the initial data). *Assume that the functions*

$$K_X^\pm(\lambda, \mu) = B_X(\lambda, \mu) + 2\partial_\lambda B_X(\lambda, \mu)(\lambda \pm \sqrt{\lambda\mu})$$

are bounded from below:

$$k_2 := \inf_{\pm, X \in \mathbb{Z}^n, \lambda \geq 0, \mu \geq 0} K_X^\pm(\lambda, \mu) > -\infty. \quad (\text{A.11})$$

Define

$$\tau_2 = \begin{cases} \sqrt{-1/k_2}, & k_2 < 0; \\ +\infty, & k_2 \geq 0. \end{cases}$$

Let $\tau \in (0, \tau_2)$ and $\varepsilon > 0$.

- (i) *There exists a solution to the Cauchy problem for equation (A.7) with arbitrary initial data (ψ^0, ψ^1) , and this solution is unique.*
- (ii) *For any $T > 0$, the map*

$$U(T) : (\psi^T, \psi^{T+1})|_{T=0} \mapsto (\psi^T, \psi^{T+1})$$

is continuous as a map from $l^\infty(\mathbb{Z}^n) \times l^\infty(\mathbb{Z}^n)$ to $l^\infty(\mathbb{Z}^n) \times l^\infty(\mathbb{Z}^n)$.

Remark A.3. Note that since

$$K_X^-(\lambda, \lambda) = B_X(\lambda, \lambda) = \partial_\lambda W_X(\lambda),$$

the values of k_1 and k_2 from Theorem A.1 and Theorem A.2, whether $k_2 > -\infty$, are related by $k_2 \leq k_1$, and then the values of τ_1 and τ_2 from these theorems are related by $\tau_2 \leq \tau_1$.

Theorem A.4 (Existence and uniqueness for polynomial nonlinearities).

- (i) *The condition (A.11) holds for any confining polynomial potential (A.10).*
- (ii) *Assume that*

$$V_X(\lambda) = \sum_{q=0}^4 C_{X,q} \lambda^{q+1}, \quad X \in \mathbb{Z}^n, \quad \lambda \geq 0, \quad (\text{A.12})$$

where $C_{X,q} \geq 0$ for $X \in \mathbb{Z}^n$ and $1 \leq q \leq 4$, and $C_{X,0}$ are uniformly bounded from below:

$$k_3 := \inf_{X \in \mathbb{Z}^n} C_{X,0} > -\infty. \quad (\text{A.13})$$

$$\tau_3 = \begin{cases} \sqrt{-1/k_3}, & k_3 < 0; \\ +\infty, & k_3 \geq 0. \end{cases}$$

Then for any $\tau \in (0, \tau_3)$ and any $\varepsilon > 0$ there exists a solution to the Cauchy problem for equation (A.7) with arbitrary initial data (ψ^0, ψ^1) , and this solution is unique.

Thus, even though the potential (A.10) satisfies conditions (A.9) and (A.11) in Theorem A.1 and Theorem A.2, the corresponding values τ_1 and τ_2 could be hard to specify explicitly. Yet, the second part of Theorem A.4 gives a simple description of a class of X -dependent polynomials $V_X(\lambda)$ for which the range of admissible $\tau > 0$ can be readily specified.

We will prove existence and uniqueness results stated in Theorems A.1, A.2, and A.4 in Appendix A.7.

A.4 Energy conservation

Theorem A.5 (Energy conservation). *Let ψ be a solution to equation (A.7) such that $\psi^T \in l^2(\mathbb{Z}^n)$ for all $T \in \mathbb{Z}$. Then the discrete energy*

$$E^T = \sum_{X \in \mathbb{Z}^n} \varepsilon^n \left[\left(\frac{1}{\tau^2} - \frac{n}{\varepsilon^2} \right) \frac{|\psi_X^{T+1} - \psi_X^T|^2}{2} + \sum_{j=1}^n \sum_{\pm} \frac{|\psi_X^{T+1} - \psi_{X \pm e_j}^T|^2}{4\varepsilon^2} + \frac{V_X(|\psi_X^{T+1}|^2) + V_X(|\psi_X^T|^2)}{2} \right] \quad (\text{A.14})$$

is conserved.

Remark A.6. The discrete energy is positive-definite if the grid ratio satisfies

$$\frac{\tau}{\varepsilon} \leq \frac{1}{\sqrt{n}}. \quad (\text{A.15})$$

Remark A.7. If ψ^0 and $\psi^1 \in l^2(\mathbb{Z}^n)$, then, by Theorem A.1, one also has $\psi^T \in l^2(\mathbb{Z}^n)$ for all $T \in \mathbb{Z}$ as long as

$$\inf_{X \in \mathbb{Z}^n, \lambda \geq 0} \partial_\lambda V_X(\lambda) > -\infty.$$

Proof. For any $u, v \in \mathbb{C}$, there is the identity

$$|u|^2 - |v|^2 = \operatorname{Re} [(\bar{u} - \bar{v}) \cdot (u + v)]. \quad (\text{A.16})$$

Applying (A.16), one has:

$$\sum_{X \in \mathbb{Z}^n} (|\psi_X^{T+1} - \psi_X^T|^2 - |\psi_X^T - \psi_X^{T-1}|^2) = \operatorname{Re} \sum_{X \in \mathbb{Z}^n} (\bar{\psi}_X^{T+1} - \bar{\psi}_X^{T-1}) \cdot (\psi_X^{T+1} - 2\psi_X^T + \psi_X^{T-1}). \quad (\text{A.17})$$

Using (A.16), we also derive the following identity for any function $\psi_X^T \in \mathbb{C}$:

$$\begin{aligned} & \sum_{X \in \mathbb{Z}^n} \sum_{j=1}^n \left[|\psi_X^{T+1} - \psi_{X-e_j}^T|^2 - |\psi_{X-e_j}^T - \psi_X^{T-1}|^2 \right. \\ & \quad \left. + |\psi_X^{T+1} - \psi_{X+e_j}^T|^2 - |\psi_{X+e_j}^T - \psi_X^{T-1}|^2 \right] \\ &= \operatorname{Re} \sum_{X \in \mathbb{Z}^n} \sum_{j=1}^n \left[(\bar{\psi}_X^{T+1} - \bar{\psi}_X^{T-1}) \cdot (\psi_X^{T+1} - 2\psi_{X \pm e_j}^T + \psi_X^{T-1}) \right. \\ & \quad \left. + (\bar{\psi}_X^{T+1} - \bar{\psi}_X^{T-1}) \cdot (\psi_X^{T+1} - 2\psi_{X+e_j}^T + \psi_X^{T-1}) \right] \\ &= \operatorname{Re} \sum_{X \in \mathbb{Z}^n} (\bar{\psi}_X^{T+1} - \bar{\psi}_X^{T-1}) \cdot \left[2n(\psi_X^{T+1} - 2\psi_X^T + \psi_X^{T-1}) \right. \\ & \quad \left. - 2 \sum_{j=1}^n (\psi_{X+e_j}^T - 2\psi_X^T + \psi_{X-e_j}^T) \right]. \end{aligned} \quad (\text{A.18})$$

Further, (A.6) together with (A.16) imply that

$$\begin{aligned} & V_X(|\psi_X^{T+1}|^2) - V_X(|\psi_X^{T-1}|^2) \\ &= \operatorname{Re} [(\bar{\psi}_X^{T+1} - \bar{\psi}_X^{T-1}) \cdot (\psi_X^{T+1} + \psi_X^{T-1})] B_X(|\psi_X^{T+1}|^2, |\psi_X^{T-1}|^2). \end{aligned} \quad (\text{A.19})$$

Taking into account (A.17), (A.18), and (A.19), we compute:

$$\begin{aligned}
\frac{E^T - E^{T-1}}{\varepsilon^n} &= \sum_{X \in \mathbb{Z}^n} \left[\left(\frac{1}{\tau^2} - \frac{n}{\varepsilon^2} \right) \frac{|\psi_X^{T+1} - \psi_X^T|^2 - |\psi_X^T - \psi_X^{T-1}|^2}{2} \right. \\
&\quad + \sum_{j=1}^n \sum_{\pm} \frac{|\psi_X^{T+1} - \psi_{X \pm e_j}^T|^2 - |\psi_{X \pm e_j}^T - \psi_X^{T-1}|^2}{4\varepsilon^2} \\
&\quad \left. + \frac{V_X(|\psi_X^{T+1}|^2) - V_X(|\psi_X^{T-1}|^2)}{2} \right] \\
&= \operatorname{Re} \sum_{X \in \mathbb{Z}^n} (\bar{\psi}_X^{T+1} - \bar{\psi}_X^{T-1}) \cdot \left[\left(\frac{1}{\tau^2} - \frac{n}{\varepsilon^2} \right) \frac{\psi_X^{T+1} - 2\psi_X^T + \psi_X^{T-1}}{2} \right. \\
&\quad + \frac{n(\psi_X^{T+1} - 2\psi_X^T + \psi_X^{T-1}) - \sum_{j=1}^n (\psi_{X+e_j}^T - 2\psi_X^T + \psi_{X-e_j}^T)}{2\varepsilon^2} \\
&\quad \left. + \frac{\psi_X^{T+1} + \psi_X^{T-1}}{2} B_X(|\psi_X^{T+1}|^2, |\psi_X^{T-1}|^2) \right].
\end{aligned}$$

The expression in the square brackets adds up to zero due to (A.7). We conclude that $E^T = E^{T-1}$ for all $T \in \mathbb{Z}$. \square

A.5 A priori estimates

Theorem A.8 (A priori estimates). *Assume that $\varepsilon > 0$ and $\tau > 0$ satisfy*

$$\frac{\tau}{\varepsilon} \leq \frac{1}{\sqrt{n}}.$$

Assume that

$$V_X(\lambda) = \frac{m^2}{2}\lambda + W_X(\lambda), \quad (\text{A.20})$$

where $m > 0$, and for each $X \in \mathbb{Z}^n$ the function $W_X \in C^2(\mathbb{R})$ satisfies $W_X(\lambda) \geq 0$ for $\lambda \geq 0$. Then any solution ψ_X^T to the Cauchy problem (A.7) with arbitrary initial data $(\psi^0, \psi^1) \in l^2(\mathbb{Z}^n) \times l^2(\mathbb{Z}^n)$ satisfies the a priori estimate

$$\varepsilon^n \|\psi^T\|_{l^2}^2 \leq \frac{4E^0}{m^2}, \quad (\text{A.21})$$

where E^0 is the energy (A.14) of the solution ψ_X^T at the moment $T = 0$.

Proof. This immediately follows from the conservation of the energy (A.14) with $V_X(\lambda)$ given by (A.20),

$$\begin{aligned}
E^T &= \sum_{X \in \mathbb{Z}^n} \varepsilon^n \left[\left(\frac{1}{\tau^2} - \frac{n}{\varepsilon^2} \right) \frac{|\psi_X^{T+1} - \psi_X^T|^2}{2} + \sum_{j=1}^n \frac{|\psi_X^{T+1} - \psi_{X-e_j}^T|^2 + |\psi_X^{T+1} - \psi_{X+e_j}^T|^2}{4\varepsilon^2} \right. \\
&\quad \left. + \frac{m^2(|\psi_X^{T+1}|^2 + |\psi_X^T|^2)}{4} + \frac{W_X(|\psi_X^{T+1}|^2) + W_X(|\psi_X^T|^2)}{2} \right].
\end{aligned}$$

\square

Remark A.9. In the continuous limit $\varepsilon \rightarrow 0$, the relation (A.21) is similar to the a priori estimate (A.5) for the solutions to the continuous nonlinear Klein-Gordon equation (A.4).

Remark A.10. In [SV78], in the case $\psi_X^T \in \mathbb{R}$, $(X, T) \in \mathbb{Z} \times \mathbb{Z}$ (in the dimension $n = 1$), the following expression for the discretized energy was introduced:

$$\begin{aligned}
E_{SV}^T &= \frac{1}{2} \sum_{X \in \mathbb{Z}^n} \left[\frac{(\psi_X^{T+1} - \psi_X^T)^2}{\tau^2} + \frac{(\psi_{X+1}^{T+1} - \psi_X^{T+1})(\psi_{X+1}^T - \psi_X^T)}{\varepsilon^2} \right. \\
&\quad \left. + V(|\psi_X^{T+1}|^2) + V(|\psi_X^T|^2) \right].
\end{aligned} \quad (\text{A.22})$$

The presence of the second term which is not positive-definite deprives one of the a priori l^2 bound on ψ , such as the one stated in Theorem A.8. In view of this, the Strauss-Vazquez finite-difference scheme for the nonlinear Klein-Gordon equation is not *unconditionally stable*. Other schemes (conditionally and unconditionally stable) were proposed in [LVQ95, Fur01]. Now, due to the a priori bound (A.21), we deduce that, as the matter of fact, the Strauss-Vazquez scheme is stable in n dimensions under the condition that the grid ratio is $\tau/\varepsilon \leq 1/\sqrt{n}$. Note that in the case $\psi \in \mathbb{R}$, the Strauss-Vazquez energy (A.22) agrees with the energy defined in (A.14).

A.6 The charge conservation

Let us consider the charge conservation. We will define the discrete charge under the following assumption:

Assumption A.11.

$$\frac{\tau}{\varepsilon} = \frac{1}{\sqrt{n}}.$$

Under Assumption A.11, ψ_X^T drops out of equation (A.7); the latter can be written as

$$(\psi_X^{T+1} + \psi_X^{T-1})(1 + \tau^2 B_X(|\psi_X^{T+1}|^2, |\psi_X^{T-1}|^2)) = \frac{1}{n} \sum_{j=1}^n (\psi_{X+e_j}^T + \psi_{X-e_j}^T). \quad (\text{A.23})$$

Theorem A.12 (Charge conservation). *Let Assumption A.11 be satisfied. Let ψ be a solution to equation (A.23) such that $\psi^T \in l^2(\mathbb{Z}^n)$ for all $T \in \mathbb{Z}$ (see Theorem A.1). Then the discrete charge*

$$Q^T = \frac{i}{4\tau} \sum_{X \in \mathbb{Z}^n} \varepsilon^n [\bar{\psi}_{X+e_j}^T \cdot \psi_X^{T+1} + \bar{\psi}_{X-e_j}^T \cdot \psi_X^{T+1} - \bar{\psi}_X^{T+1} \cdot \psi_{X+e_j}^T - \bar{\psi}_X^{T+1} \cdot \psi_{X-e_j}^T] \quad (\text{A.24})$$

is conserved.

Remark A.13. The continuous limit of the discrete charge Q defined in (A.24) coincides with the charge functional (A.3) of the continuous nonlinear wave equation (A.1).

Proof. Let us prove the charge conservation. One has:

$$\begin{aligned} \frac{4\tau}{i\varepsilon^n} Q^T &= \sum_{X \in \mathbb{Z}^n} \sum_{j=1}^n \sum_{\pm} \left[\bar{\psi}_{X \pm e_j}^T \cdot \psi_X^{T+1} - \text{c. c.} \right], \\ \frac{4\tau}{i\varepsilon^n} Q^{T-1} &= \sum_{X \in \mathbb{Z}^n} \sum_{j=1}^n \sum_{\pm} \left[\bar{\psi}_{X \pm e_j}^{T-1} \cdot \psi_X^T - \text{c. c.} \right] = - \sum_{X \in \mathbb{Z}^n} \sum_{j=1}^n \sum_{\pm} \left[\bar{\psi}_{X \pm e_j}^T \cdot \psi_X^{T-1} - \text{c. c.} \right], \end{aligned}$$

where c. c. denotes the complex conjugation of the preceding expression. Therefore,

$$\begin{aligned} \frac{4\tau(Q^T - Q^{T-1})}{i\varepsilon^n} &= \sum_{X \in \mathbb{Z}^n} \sum_{j=1}^n \sum_{\pm} \bar{\psi}_{X \pm e_j}^T \cdot (\psi_X^{T+1} + \psi_X^{T-1}) - \text{c. c.} \\ &= n \sum_{X \in \mathbb{Z}^n} (1 + \tau^2 B_X(|\psi_X^{T+1}|^2, |\psi_X^{T-1}|^2)) |\psi_X^{T+1} + \psi_X^{T-1}|^2 - \text{c. c.} = 0. \end{aligned}$$

To get to the second line, we used the complex conjugate of (A.23). This finishes the proof of Theorem A.12. \square

A.7 Proof of well-posedness

First, we prove the existence.

Proof of Theorem A.1. We rewrite equation (A.7) in the following form:

$$\begin{aligned} &(\psi_X^{T+1} + \psi_X^{T-1})(1 + \tau^2 B_X(|\psi_X^{T+1}|^2, |\psi_X^{T-1}|^2)) \\ &= \frac{\tau^2}{\varepsilon^2} \sum_{j=1}^n (\psi_{X+e_j}^T - 2\psi_X^T + \psi_{X-e_j}^T) + 2\psi_X^T, \quad X \in \mathbb{Z}^n, \quad T \in \mathbb{Z}. \end{aligned} \quad (\text{A.25})$$

By (A.9) and the choice of τ_1 in Theorem A.1, for $\tau \in (0, \tau_1)$ one has

$$\inf_{X \in \mathbb{Z}^n, \lambda \geq 0} (1 + \tau^2 \partial_\lambda V_X(\lambda)) > 0. \quad (\text{A.26})$$

Since

$$\inf_{\substack{X \in \mathbb{Z}^n \\ \lambda \geq 0, \mu \geq 0}} B_X(\lambda, \mu) = \inf_{\substack{X \in \mathbb{Z}^n \\ \lambda \geq 0, \mu \geq 0, \lambda \neq \mu}} \frac{V_X(\lambda) - V_X(\mu)}{\lambda - \mu} = \inf_{X \in \mathbb{Z}^n, \lambda \geq 0} \partial_\lambda V_X(\lambda), \quad (\text{A.27})$$

inequality (A.26) yields

$$c := \inf_{X \in \mathbb{Z}^n, \lambda \geq 0, \mu \geq 0} (1 + \tau^2 B_X(\lambda, \mu)) > 0. \quad (\text{A.28})$$

Let us show that equation (A.25) allows us to find ψ_X^{T+1} , for any given $X \in \mathbb{Z}^n$ and $T \in \mathbb{Z}$, once one knows ψ^T and ψ^{T-1} . Equation (A.25) implies that

$$(1 + \tau^2 B_X(|\psi_X^{T+1}|^2, |\psi_X^{T-1}|^2))(\psi_X^{T+1} + \psi_X^{T-1}) = \xi_X^T, \quad (\text{A.29})$$

$$\xi_X^T := \frac{\tau^2}{\varepsilon^2} \sum_{j=1}^n (\psi_{X+e_j}^T - 2\psi_X^T + \psi_{X-e_j}^T) + 2\psi_X^T \in \mathbb{C}. \quad (\text{A.30})$$

If $\xi_X^T = 0$, then there is a solution to (A.29) given by $\psi_X^{T+1} = -\psi_X^{T-1}$. Due to (A.28), this solution is unique. Now let us assume that $\xi_X^T \neq 0$. We see from (A.29) that we are to have

$$\psi_X^{T+1} + \psi_X^{T-1} = s \xi_X^T, \quad \text{with some } s \in \mathbb{R}. \quad (\text{A.31})$$

Let us introduce the function

$$f(s) := (1 + \tau^2 B_X(|s \xi_X^T - \psi_X^{T-1}|^2, |\psi_X^{T-1}|^2))s. \quad (\text{A.32})$$

We do not indicate dependence of f on ψ_X^{T-1} , ξ_X^T , and X , treating them as parameters. For $\xi_X^T \neq 0$, we can solve (A.29) if we can find $s \in \mathbb{R}$ such that

$$f(s) = 1. \quad (\text{A.33})$$

Since $f(0) = 0$, while $\lim_{s \rightarrow \infty} f(s) = +\infty$ by (A.28), one concludes that there is at least one solution $s > 0$ to (A.33).

Let us prove that once $(\psi^0, \psi^1) \in l^2(\mathbb{Z}^n) \times l^2(\mathbb{Z}^n)$, then one also knows that $\|\psi^T\|_{l^2(\mathbb{Z}^n)}$ remains finite (but not necessarily uniformly bounded) for all $T \in \mathbb{Z}$. As it follows from (A.28) and (A.29),

$$|\psi_X^{T+1}| \leq \frac{1}{c} |\xi_X^T| + |\psi_X^{T-1}|. \quad (\text{A.34})$$

Since $\|\xi^T\|_{l^2(\mathbb{Z}^n)} \leq \left(\frac{4\tau^2}{\varepsilon^2} + 2\right) \|\psi^T\|_{l^2(\mathbb{Z}^n)}$ by (A.30), the relation (A.34) implies the estimate

$$\|\psi^{T+1}\|_{l^2(\mathbb{Z}^n)} \leq \frac{1}{c} \left(\frac{4\tau^2}{\varepsilon^2} + 2\right) \|\psi^T\|_{l^2(\mathbb{Z}^n)} + \|\psi^{T-1}\|_{l^2(\mathbb{Z}^n)}, \quad (\text{A.35})$$

and, by recursion, the finiteness of $\|\psi^T\|_{l^2(\mathbb{Z}^n)}$ for all $T \geq 0$. The case $T \leq 0$ is finished in the same way. \square

Now we prove the uniqueness of solutions to the Cauchy problem for equation (A.7) and the continuous dependence on the initial data.

Proof of Theorem A.2. First, note that, by Remark A.3, τ_1 from Theorem A.1 and τ_2 from Theorem A.2 are related by $\tau_2 \leq \tau_1$. Therefore, the existence of a solution ψ_X^T to the Cauchy problem for equation (A.7) follows from Theorem A.1.

Let us prove that this solution ψ_X^T is unique. When in (A.30) one has

$$\xi_X^T := \frac{\tau^2}{\varepsilon^2} \sum_{j=1}^n (\psi_{X+e_j}^T - 2\psi_X^T + \psi_{X-e_j}^T) + 2\psi_X^T = 0,$$

then, by (A.28), the only solution ψ_X^{T+1} to (A.29) is given by $\psi_X^{T+1} = -\psi_X^{T-1}$. We now consider the case $\xi_X^T \neq 0$. By (A.29), (A.31), and (A.32), it suffices to prove the uniqueness of the solution to (A.33). This will follow if we show that $f(s)$ satisfies

$$f'(s) > 0, \quad s \in \mathbb{R}. \quad (\text{A.36})$$

The explicit expression for $f'(s)$ is

$$\begin{aligned} f'(s) &= 1 + \tau^2 B_X(|s\xi_X^T - \psi_X^{T-1}|^2, |\psi_X^{T-1}|^2) \\ &\quad + \tau^2 \partial_\lambda B_X(|s\xi_X^T - \psi_X^{T-1}|^2, |\psi_X^{T-1}|^2) (-2\operatorname{Re}(\bar{\psi}_X^{T-1} \cdot \xi_X^T) + 2|\xi_X^T|^2 s). \end{aligned} \quad (\text{A.37})$$

Using the relation (A.31), we derive the identity

$$\begin{aligned} (-2\operatorname{Re}(\bar{\psi}_X^{T-1} \cdot \xi_X^T) + 2|\xi_X^T|^2 s) &= 2 \left| s\xi_X^T - \frac{\psi_X^{T-1}}{2} \right|^2 - \frac{|\psi_X^{T-1}|^2}{2} \\ &= 2 \left| \psi_X^{T+1} + \frac{\psi_X^{T-1}}{2} \right|^2 - \frac{|\psi_X^{T-1}|^2}{2} \end{aligned}$$

and rewrite the expression (A.37) for $f'(s)$ as

$$\begin{aligned} f'(s) &= 1 + \tau^2 \left[B_X(|\psi_X^{T+1}|^2, |\psi_X^{T-1}|^2) \right. \\ &\quad \left. + 2\partial_\lambda B_X(|\psi_X^{T+1}|^2, |\psi_X^{T-1}|^2) \left(|\psi_X^{T+1} + \frac{\psi_X^{T-1}}{2}|^2 - \frac{|\psi_X^{T-1}|^2}{4} \right) \right]. \end{aligned} \quad (\text{A.38})$$

We denote $\lambda = |\psi_X^{T+1}|^2$, $\mu = |\psi_X^{T-1}|^2$. Since

$$\lambda - \sqrt{\lambda\mu} + \frac{\mu}{4} \leq |\psi_X^{T+1} + \frac{\psi_X^{T-1}}{2}|^2 \leq \lambda + \sqrt{\lambda\mu} + \frac{\mu}{4},$$

we see that

$$f'(s) \geq 1 + \tau^2 \min_{\pm} \inf_{X \in \mathbb{Z}^n, \lambda \geq 0, \mu \geq 0} K_X^\pm(\lambda, \mu), \quad (\text{A.39})$$

where

$$K_X^\pm(\lambda, \mu) = B_X(\lambda, \mu) + 2\partial_\lambda B_X(\lambda, \mu)(\lambda \pm \sqrt{\lambda\mu}).$$

By (A.11) and by our choice of τ_2 in Theorem A.2, for any $\tau \in (0, \tau_2)$ we have

$$\kappa := \inf_{X \in \mathbb{Z}^n, \lambda \geq 0, \mu \geq 0} \{1 + \tau^2 K_X^\pm(\lambda, \mu)\} > 0;$$

then, by (A.39), $f'(s) \geq \kappa$, where $\kappa > 0$. It follows that for $\xi_X^T \neq 0$ there is a unique solution s to (A.33); moreover, this solution s continuously depends on ψ_X^{T-1} and on ξ_X^T . The latter in turn continuously depends on $\psi_{X \pm e_j}^T$, $1 \leq j \leq n$ (Cf. (A.30)). Thus, by (A.31), the solution ψ_X^{T+1} to equation (A.29) is uniquely defined and continuously depends on ψ_X^{T-1} and $\psi_{X \pm e_j}^T$, $1 \leq j \leq n$.

This finishes the proof of Theorem A.2. \square

Proof of Theorem A.4. Let us prove that the condition (A.11) in Theorem A.2 is satisfied by any polynomial potential of the form (A.10). The inequality (A.11) will be satisfied if the highest order term from $V(\lambda)$ contributes a strictly positive expression. More precisely, we need to prove the following result.

Lemma A.14. *Let $V(\lambda) = \lambda^{p+1}$, so that $B(\lambda, \mu) = \frac{\lambda^{p+1} - \mu^{p+1}}{\lambda - \mu}$, $p \geq 0$. Then the following inequality takes place:*

$$\inf_{\lambda \geq 0, \mu \geq 0, \lambda^2 + \mu^2 = 1} \left[B(\lambda, \mu) + 2\partial_\lambda B(\lambda, \mu)(\lambda \pm \sqrt{\lambda\mu}) \right] > 0. \quad (\text{A.40})$$

Proof. Since B and $\partial_\lambda B$ are strictly positive for $\lambda^2 + \mu^2 > 0$, the inequality (A.40) is nontrivial only for the negative sign in (A.40) and only when $\mu > \lambda$. First we note that

$$B(\lambda, \mu) = \frac{\mu^{p+1} - \lambda^{p+1}}{\mu - \lambda},$$

$$\partial_\lambda B(\lambda, \mu) = \frac{-(p+1)\lambda^p(\mu - \lambda) - \lambda^{p+1} + \mu^{p+1}}{(\mu - \lambda)^2} = \frac{\mu^{p+1} - (p+1)\lambda^p\mu + p\lambda^{p+1}}{(\mu - \lambda)^2}.$$

Let $z \geq 0$ be such that $z^2 = \lambda/\mu$. To prove the lemma, we need to check that

$$\frac{1 - z^{2p+2}}{1 - z^2} + 2 \frac{1 - (p+1)z^{2p} + pz^{2p+2}}{(1 - z^2)^2} (z^2 - z) > 0, \quad 0 \leq z < 1, \quad (\text{A.41})$$

or equivalently,

$$(1+z)(1 - z^{2p+2}) - 2z(1 - (p+1)z^{2p} + pz^{2p+2}) > 0.$$

The left-hand side takes the form

$$\begin{aligned} & (1+z)(1 - z^{2p+2}) - 2z(1 - z^{2p+2} - (p+1)(z^{2p} - z^{2p+2})) \\ &= (1-z)(1 - z^{2p+2}) + 2z(p+1)(z^{2p} - z^{2p+2}), \end{aligned}$$

which is clearly strictly positive for all $0 \leq z < 1$ and $p \geq 0$, proving (A.41). \square

This finishes the proof of the first part of Theorem A.4; now we turn to the second part.

Lemma A.15 (Uniqueness criterion). *Assume that for a particular $\tau > 0$ and for all $\lambda \geq 0$, $\mu \geq 0$, $X \in \mathbb{Z}^n$, the following inequalities hold:*

$$1 + \tau^2 \inf_{X \in \mathbb{Z}^n, \lambda \geq 0, \mu \geq 0} \left(B_X(\lambda, \mu) - \partial_\lambda B_X(\lambda, \mu) \frac{\mu}{2} \right) > 0; \quad (\text{A.42})$$

$$\inf_{X \in \mathbb{Z}^n, \lambda \geq 0, \mu \geq 0} \partial_\lambda B_X(\lambda, \mu) \geq 0. \quad (\text{A.43})$$

Then there is a solution ψ_X^T to the Cauchy problem for equation (A.7) with arbitrary initial data (ψ^0, ψ^1) , and this solution is unique.

Proof of Lemma A.15. The inequalities (A.42) and (A.43) lead to

$$1 + \tau^2 \inf_{X \in \mathbb{Z}^n, \lambda \geq 0} B_X(\lambda, \lambda) > 0,$$

hence, by the same argument as in Theorem A.1, there is a solution ψ_X^T . The relation (A.38) shows that $f'(s) \geq c$ for some $c > 0$. The rest of the proof is the same as for Theorem A.2. \square

In the second part of Theorem A.4, we assume that

$$V_X(\lambda) = \sum_{q=0}^4 C_{X,q} \lambda^{q+1}, \quad X \in \mathbb{Z}^n, \quad \lambda \geq 0, \quad (\text{A.44})$$

where $C_{X,q} \geq 0$ for $X \in \mathbb{Z}^n$ and $1 \leq q \leq 4$, and

$$k_3 = \inf_{X \in \mathbb{Z}^n} C_{X,0} > -\infty. \quad (\text{A.45})$$

Thus, the term $C_{X,0}\lambda$ in $V_X(\lambda)$ contributes to $B_X(\lambda, \mu)$ the expression $b_{X,0}(\lambda, \mu) = C_{X,0}$, while each term in $V_X(\lambda)$ of the form $C_{X,q}\lambda^{q+1}$, with $1 \leq q \leq 4$ and $C_{X,q} \geq 0$, contributes to $B_X(\lambda, \mu)$ the expression $C_{X,q}b_q(\lambda, \mu)$, with $b_q(\lambda, \mu) = \sum_{k=0}^q \lambda^{q-k}\mu^k$. For $\tau \in (0, \tau_3)$, with $\tau_3 = \sqrt{-1/k_3}$ for $k_3 < 0$ and $\tau_3 = +\infty$ for $k_3 \geq 0$, one has

$$1 + \tau^2 \inf_{X \in \mathbb{Z}^n} C_{X,0} > 0. \quad (\text{A.46})$$

Lemma A.16. *For $1 \leq q \leq 4$, $b_q(\lambda, \mu) = \sum_{k=0}^q \lambda^{q-k}\mu^k$ satisfies the inequality*

$$b_q(\lambda, \mu) \geq \partial_\lambda b_q(\lambda, \mu) \frac{\mu}{2} \quad \text{for all } \lambda, \mu \geq 0.$$

By (A.46) and Lemma A.16, condition (A.42) is satisfied. Since $C_{X,q} \geq 0$ for $1 \leq q \leq 4$, each term $C_{X,q}b_q(\lambda, \mu)$ satisfies condition (A.43). Therefore, by Lemma A.15, there is a unique solution ψ_X^T to the Cauchy problem for equation (A.7). This finishes the proof of Theorem A.4. \square

B Titchmarsh convolution theorem for distributions on the circle

The Titchmarsh convolution theorem [Tit26] states that for any two compactly supported distributions $f, g \in \mathcal{E}'(\mathbb{R})$,

$$\inf \text{supp } f * g = \inf \text{supp } f + \inf \text{supp } g, \quad \sup \text{supp } f * g = \sup \text{supp } f + \sup \text{supp } g. \quad (\text{B.1})$$

The higher-dimensional reformulation by Lions [Lio51] states that for $f, g \in \mathcal{E}'(\mathbb{R}^n)$, the convex hull of the support of $f * g$ is equal to the sum of convex hulls of supports of f and g . Different proofs of the Titchmarsh convolution theorem are contained in [Yos80, Chapter VI] (Real Analysis style), [Hör90, Theorem 4.3.3] (Harmonic Analysis style), and [Lev96, Lecture 16, Theorem 5] (Complex Analysis style). Here we give a version of the Titchmarsh Theorem which is valid for distributions supported in $n > 1$ small intervals of the circle $\mathbb{T} = \mathbb{R} \bmod 2\pi$. For brevity, we only give the result for distributions supported in two small intervals, which suffices for the applications in this paper; a more general version is proved in [KK12].

First, we note that there are zero divisors with respect to the convolution on the circle. Indeed, for any two distributions $f, g \in \mathcal{E}'(\mathbb{T})$ one has

$$(f + S_\pi f) * (g - S_\pi g) = f * g + S_\pi(f * g) - S_\pi(f * g) - f * g = 0. \quad (\text{B.2})$$

Above, $S_y, y \in \mathbb{T}$, is the shift operator, defined on $\mathcal{E}'(\mathbb{T})$ by

$$(S_y f)(\omega) = f(\omega - y), \quad (\text{B.3})$$

where the above relation is understood in the sense of distributions. Yet, the cases when the Titchmarsh convolution theorem “does not hold” (in a certain naïve form) could be specified. This leads to a version of the Titchmarsh convolution theorem for distributions on the circle (Theorem B.1 below).

Let us start with the following problem which illustrates our methods.

Problem. Let $f, g \in \mathcal{E}'(\mathbb{R})$ be such that $\inf \text{supp } f = 0, \inf \text{supp } g = 0$. If $(f * f)|_{(-\infty, A)} = (g * g)|_{(-\infty, A)}$, for some $A > 0$, then on $(-\infty, A)$ either $f = g$, or $f = -g$.

Solution. Since $\inf \text{supp}(f - g) * (f + g) = \inf \text{supp}(f * f - g * g) \geq A$, the Titchmarsh convolution theorem (B.1) states that

$$\inf \text{supp}(f - g) + \inf \text{supp}(f + g) \geq A. \quad (\text{B.4})$$

One concludes that either $\inf \text{supp}(f - g) \geq A/2$, or $\inf \text{supp}(f + g) \geq A/2$. In the former case, we have $f = g$ on $(-\infty, A/2)$, hence $\inf \text{supp}(f + g) = \inf \text{supp } f = 0$. Therefore (B.4) shows that $\inf \text{supp}(f - g) \geq A$, hence $f = g$ on $(-\infty, A)$. Similarly, if $\inf \text{supp}(f + g) \geq A/2$, one concludes that $\inf \text{supp}(f + g) \geq A$, thus $f = -g$ on $(-\infty, A)$.

For $I \subset \mathbb{T}$, denote

$$\mathcal{R}_2(I) = \bigcup_{k \in \mathbb{Z}_2} S_{\pi k} I, \quad \text{where } \mathbb{Z}_2 = \mathbb{Z} \bmod 2.$$

Theorem B.1 (Titchmarsh theorem for distributions on the circle). *Let $f, g \in \mathcal{E}'(\mathbb{T})$. Let $I, J \subset \mathbb{T}$ be two closed intervals such that $\text{supp } f \subset \mathcal{R}_2(I)$, $\text{supp } g \subset \mathcal{R}_2(J)$, and assume that there is no closed interval $I' \subsetneq I$ such that $\text{supp } f \subset \mathcal{R}_2(I')$ and no closed interval $J' \subsetneq J$ such that $\text{supp } g \subset \mathcal{R}_2(J')$.*

Assume that

$$|I| + |J| < \pi. \quad (\text{B.5})$$

*Let $K \subset I + J \subset \mathbb{T}$ be a closed interval such that $\text{supp } f * g \subset \mathcal{R}_2(K)$. Then $\lambda := \inf K - \inf I - \inf J > 0$ if and only if there is $\sigma \in \{\pm 1\}$ such that*

$$(f + \sigma S_\pi f) \Big|_{(\sup I - \pi, \inf I + \lambda)} = 0, \quad (g - \sigma S_\pi g) \Big|_{(\sup J - \pi, \inf J + \lambda)} = 0. \quad (\text{B.6})$$

Similarly, one has $\rho := \sup I + \sup J - \sup K > 0$ if and only if there is $\sigma \in \{\pm 1\}$ such that

$$(f + \sigma S_\pi f) \Big|_{(\sup I - \rho, \inf I + \pi)} = 0, \quad (g - \sigma S_\pi g) \Big|_{(\sup J - \rho, \inf J + \pi)} = 0. \quad (\text{B.7})$$

Remark B.2. While $\mathcal{E}'(\mathbb{T}) = \mathcal{D}'(\mathbb{T})$, we use the notation $\mathcal{E}'(\mathbb{T})$ for the consistency with the requirements of the standard Titchmarsh convolution theorem (B.1).

Informally, we could say that the intervals I , J , and K play the role similar to “convex hulls” of supports of f , g , $f * g \in \mathcal{E}'(\mathbb{T})$. If $K \subsetneq I + J$ (certain “naïve form of the Titchmarsh convolution theorem” is not satisfied), then both f and g satisfy certain symmetry properties on $\mathcal{R}_2(U)$ and on $\mathcal{R}_2(V)$, where open non-intersecting intervals U and V can be chosen so that $U \cup K \cup V \supset I + J$.

Proof of Theorem B.1. We will only prove (B.6); the relations (B.7) follow by applying the reflection to \mathbb{T} and using the first part of the theorem.

The “if” part of the theorem is checked by direct computation. Let $f \in \mathcal{E}'(I \cup S_\pi I)$, where $I \subset \mathbb{T}$, $|I| < \pi/2$, $g \in \mathcal{E}'(J \cup S_\pi J)$, where $J \subset \mathbb{T}$, $|J| < \pi/2$, and assume that $f = \pm S_\pi f$ on $(\sup I - \pi, \inf I + \lambda)$, $g = \mp S_\pi g$ on $(\sup J - \pi, \inf J + \lambda)$. Then, as in (B.2),

$$\begin{aligned} & (f * g)|_{(\sup I + \sup J - 2\pi, \inf I + \inf J + \lambda)} \\ &= f|_{(\sup I - \pi, \inf I + \lambda)} * g|_{(\sup J - \pi, \inf J + \lambda)} + (S_\pi f)|_{(\sup I - \pi, \inf I + \lambda)} * (S_\pi g)|_{(\sup J - \pi, \inf J + \lambda)} \\ &= f|_{(\sup I - \pi, \inf I + \lambda)} * g|_{(\sup J - \pi, \inf J + \lambda)} - f|_{(\sup I - \pi, \inf I + \lambda)} * g|_{(\sup J - \pi, \inf J + \lambda)} = 0. \end{aligned}$$

Let us now prove the “only if” part. One has $\text{supp } f \subset \mathcal{R}_2(I)$, $\text{supp } g \subset \mathcal{R}_2(J)$, $\text{supp } f * g \subset \mathcal{R}_2(K) \subset \mathcal{R}_2(I + J)$. Due to the restriction (B.5), the sets $\mathcal{R}_2(I)$, $\mathcal{R}_2(J)$, and $\mathcal{R}_2(I + J)$ each consist of n non-intersecting intervals. For $j \in \mathbb{Z}_2$, let us set $f_j = (S_{\pi j} f)|_I \in \mathcal{E}'(I)$, $g_j = (S_{\pi j} g)|_J \in \mathcal{E}'(J)$, $h_j = (S_{\pi j} (f * g))|_K \in \mathcal{E}'(I + J)$; then, for $j \in \mathbb{Z}_2$,

$$h_j = (S_{\pi j} (f * g))|_{I+J} = \sum_{\substack{k+l=j \pmod{2} \\ k, l \in \mathbb{Z}_2}} (S_{\pi k} f)|_I * (S_{\pi l} g)|_J = \sum_{\substack{k+l=j \pmod{2} \\ k, l \in \mathbb{Z}_2}} f_k * g_l. \quad (\text{B.8})$$

Using the relation (B.8), for $\sigma = \pm 1$ we have:

$$(f_0 + \sigma f_1) * (g_0 + \sigma g_1) = (f_0 * g_0 + f_1 * g_1) + \sigma(f_0 * g_1 + f_1 * g_0) = h_0 + \sigma h_1. \quad (\text{B.9})$$

Applying the Titchmarsh convolution theorem (B.1) to this relation, we obtain:

$$\inf \text{supp}(f_0 + \sigma f_1) + \inf \text{supp}(g_0 + \sigma g_1) = \inf \text{supp}(h_0 + \sigma h_1) \geq \inf K, \quad (\text{B.10})$$

where we took into account that $\min_{j \in \mathbb{Z}_2} \inf \text{supp } h_j \geq \inf K$. Let us pick $\sigma \in \{\pm 1\}$ such that

$$\inf \text{supp}(g_0 + \sigma g_1) = \min_{j \in \mathbb{Z}_2} \inf \text{supp } g_j = \inf J. \quad (\text{B.11})$$

For this value of σ , (B.10) yields:

$$\inf \text{supp}(f_0 + \sigma f_1) \geq \inf K - \inf J = \inf I + \lambda,$$

proving the first relation in (B.6). The second relation in (B.6) follows due to the symmetric roles of f and g . The opposite sign (negative sign at σ) follows from (B.11). \square

Here is the convolution theorem for powers of a distribution.

Theorem B.3 (Titchmarsh theorem for powers of a distribution on the circle). *Let $f \in \mathcal{E}'(\mathbb{T})$. Let $I \subset \mathbb{T}$ be a closed interval such that $\text{supp } f \subset \mathcal{R}_2(I)$, and assume that there is no $I' \subsetneq I$ such that $\text{supp } f \subset \mathcal{R}_2(I')$.*

*Assume that $|I| < \frac{\pi}{p}$, for some $p \in \mathbb{N}$. Then the smallest closed interval $K \subset pI$ such that $\text{supp } f^{*p} \subset \mathcal{R}_2(K)$ is $K = pI$.*

Above, we used the notations $pI = \underbrace{I + \dots + I}_p$ and $f^{*p} = \underbrace{f * \dots * f}_p$.

Let us notice that the proof of Theorem B.3 for the case $p = 2$ immediately follows from Theorem B.1. (For example, the relations (B.6) with $f = g$ are mutually contradictory unless $\lambda = 0$.) By induction, this also gives the proof for $p = 2^N$, with any $N \in \mathbb{N}$, and then one can deduce the statement of Theorem B.3 for any $p \leq 2^N$, but under the condition $|I| < \frac{\pi}{2^N}$, which is stronger than $|I| < \frac{\pi}{p}$.

Proof of Theorem B.3. One has $\text{supp } f^{*p} \subset \mathcal{R}_2(pI)$. Due to the smallness of I , both $\mathcal{R}_2(I)$ and $\mathcal{R}_2(pI)$ are collections of n non-intersecting intervals. Define

$$f_j := (S_{\pi j} f)|_I \in \mathcal{E}'(I), \quad h_j := (S_{\pi j}(f^{*p}))|_I \in \mathcal{E}'(I).$$

Then

$$\begin{aligned} h_j &= (S_{\pi j}(f^{*p}))|_{pI} = \sum_{\substack{j_1 + \dots + j_p = j \pmod{2} \\ j_1, \dots, j_p \in \mathbb{Z}_2}} (S_{\pi j_1} f)|_I * \dots * (S_{\pi j_p} f)|_I \\ &= \sum_{\substack{j_1 + \dots + j_p = j \pmod{2} \\ j_1, \dots, j_p \in \mathbb{Z}_2}} f_{j_1} * \dots * f_{j_p}, \quad j \in \mathbb{Z}_2. \end{aligned} \quad (\text{B.12})$$

Taking into account (B.12), for $\sigma = \pm 1$ one has:

$$(f_0 + \sigma f_1)^{*p} = \sum_{j \in \mathbb{Z}_2} \sigma^j \left[\sum_{j_1 + \dots + j_p = j \pmod{2}} f_{j_1} * \dots * f_{j_p} \right] = h_0 + \sigma h_1. \quad (\text{B.13})$$

Now we apply the Titchmarsh convolution theorem to (B.13), getting

$$p \inf \text{supp}(f_0 + \sigma f_1) = \inf \text{supp}(h_0 + \sigma h_1).$$

There is $\sigma = \pm 1$ such that $\inf \text{supp}(f_0 + \sigma f_1) = \min_{j \in \mathbb{Z}_2} \inf \text{supp } f_j$; for this value of σ ,

$$p \min_{j \in \mathbb{Z}_2} \inf \text{supp } f_j = \inf \text{supp}(h_0 + \sigma h_1) \geq \min_{j \in \mathbb{Z}_2} \inf \text{supp } h_j.$$

On the other hand, (B.12) yields the inequalities $\inf \text{supp } h_j \geq p \min_{k \in \mathbb{Z}_2} \inf \text{supp } f_k$, for any $j \in \mathbb{Z}_2$. It follows that

$$\min_{j \in \mathbb{Z}_2} \inf \text{supp } h_j = p \min_{j \in \mathbb{Z}_2} \inf \text{supp } f_j$$

and similarly

$$\max_{j \in \mathbb{Z}_2} \sup \text{supp } h_j = p \max_{j \in \mathbb{Z}_2} \sup \text{supp } f_j.$$

□

Denote $f^\#(\omega) = \overline{f(-\omega)}$. Let $f \in \mathcal{E}'(\mathbb{T})$ and let $I \subset \mathbb{T}$ be a closed interval such that $\text{supp } f \subset \mathcal{R}_2(I)$. Assume that there is no closed interval $I' \subsetneq I$ such that $\text{supp } f \subset \mathcal{R}_2(I')$.

Theorem B.4. *If $I \subset (-\pi/2, \pi/2)$ and $|I| < \pi/2$, then the inclusion $\text{supp } f * f^\# \subset \{0; \pi\}$ implies that $\text{supp } f \subset \{\inf I; \sup I; \pi + \inf I; \pi + \sup I\}$. Moreover, there are distributions $\mu, \nu \in \mathcal{E}'(\mathbb{T})$, each supported at a point, such that*

$$f = \mu + S_\pi \mu + \nu - S_\pi \nu. \quad (\text{B.14})$$

Proof. If I consists of one point, $I = \{p\} \subset (-\pi/2, \pi/2)$, then $\text{supp } f = \mathcal{R}_2(p) = \{p; \pi + p\}$, and (B.14) holds with

$$\mu = \frac{f + S_\pi f}{2} \Big|_I, \quad \nu = \frac{f - S_\pi f}{2} \Big|_I.$$

Now we assume that $|I| > 0$. Define $J = -I$ and $K = \{0\} \subset I + J$. Then $\text{supp } f^\# \subset \mathcal{R}_2(J)$ and there is no $J' \subsetneq J$ such that $\text{supp } f^\# \subset \mathcal{R}_2(J')$. According to the conditions of the theorem, $\text{supp } f * f^\# \subset \mathcal{R}_2(K)$; hence, one has:

$$\lambda := \inf K - \inf I - \inf J = \sup I - \inf I = |I| > 0. \quad (\text{B.15})$$

Applying Theorem B.1 to (B.15), we conclude that there is $\sigma \in \{\pm 1\}$ such that

$$(f + \sigma S_\pi f)|_{(\sup I - \pi, \sup I)} = 0 \quad (\text{B.16})$$

and also $\inf \text{supp}(f^\# + \sigma S_\pi f^\#)|_{(-\frac{\pi}{2}, \frac{\pi}{2})} = -\sup I$; this last relation implies that

$$\sup \text{supp}(f + \sigma S_\pi f)|_{(-\frac{\pi}{2}, \frac{\pi}{2})} = \sup I. \quad (\text{B.17})$$

Also, by Theorem B.1, there is $\sigma' \in \{\pm 1\}$ such that $(f^\sharp + \sigma' S_\pi f^\sharp)|_{(-\inf I - \pi, -\inf I)} = 0$, hence

$$(f + \sigma' S_\pi f)|_{(\inf I, \inf I + \pi)} = 0. \quad (\text{B.18})$$

Comparing (B.17) with (B.18), we conclude that $\sigma' = -\sigma$; then (B.16) and (B.18) allow us to conclude that both f and $S_\pi f$ vanish on $(\inf I, \sup I)$, hence

$$\text{supp } f \subset \{\inf I; \sup I; \pi + \inf I; \pi + \sup I\}.$$

By (B.16) and (B.18), if $\sigma = 1$, the relation (B.14) holds with $\mu = f|_{(\inf I, \pi/2)}$ and $\nu = f|_{(-\pi/2, \sup I)}$. If instead $\sigma = -1$, the relation (B.14) holds with $\mu = f|_{(-\pi/2, \sup I)}$ and $\nu = f|_{(\inf I, \pi/2)}$. \square

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